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PHYSICAL REVIEW.

THE INFLUENCE OF FREQUENCY UPON THE SELF-INDUCTANCE OF CYLINDRICAL COILS OF *m*-LAYERS.

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IN the construction of an absolute standard of self-inductance for the Bureau of Standards it was found by calculation[1] that a diminution of as much as .3 per cent. in the self-inductance of the coil could take place as the frequency increased from zero to infinity.

This change, which in coils of small radius may be far larger than this, must certainly be allowed for in accurate measurements the precision of which may in certain cases reach as high as 1 in 100,000.[2]

Whether much of this change occurs at low frequencies, or whether it requires high frequencies to produce a large fraction of it, it was not possible to determine. The necessity of finding an answer to the problem suggested above, at least with reference to the calculation of standards of precision, has long been apparent to the writer.

It is true that Max Wien[3] derived an expression for the change in self-inductance of a coil of a single layer, but his results are applicable only to very low frequencies indeed, and the extension of his results, which are in the form of an infinite series, for higher frequencies, even for coils of a single layer, seems very laborious.

[1] Bull. Bureau Stand. On the Construction and Calculation of Standards of Self Inductance.

[2] Bull. Bureau Stand., Vol. I., No. 3, Rosa and Grover. Measurements of Self-Inductance.

[3] Ann. d. Physik., 14, p. 1, 1904.

A. Sommerfeld,[1] somewhat later, in a very interesting and original paper has derived an expression in elementary functions for the change in *resistance* of cylindrical coils for all frequencies. It is by means of the method originated by him that we shall be able to derive a finite expression for the change in self-inductance of a cylindrical coil of any number of layers, and valid for all frequencies. The reader is referred to his paper for the details which it was thought best not to reprint on account of length.

DERIVATION OF THE FORMULÆ.

Consider m, co-axial, cylindrical, infinitely long current-sheets, of internal and external radii

$$r_0 \quad \text{and} \quad r_1,$$
$$r_1 \quad \text{and} \quad r_2,$$
$$\cdots\cdots$$
$$r_{m-1} \quad \text{and} \quad r_m,$$

respectively. We are thus neglecting the thickness of the insulation. Let $r_1 - r_0 = r_2 - r_1 = r_3 - r_2 = \cdots = r_m - r_{m-1} = d$ be the common thickness of these current sheets.

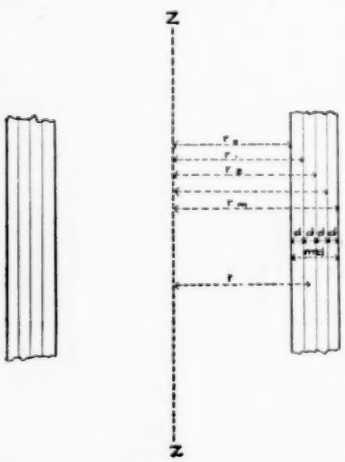

Fig. 1.

Let the symbols $S(\ )$ and $V(\ )$ denote the scalar and vector products respectively. $\nabla(\ )$ is the Hamiltonian operator. $V\nabla(\ )$ is the same as curl $(\ )$ or rot $(\ )$, while $S\nabla(\ )$ is the same as div $(\ )$.

By symmetry, the field inside of the coil is axial, and a function of "r" alone, at any given time, and on account of the infinite length of the coil, the field is all inside of the cylinder of radius r_m.

In the dielectric, Maxwell's equations are, if we neglect the dis-

[1] Ann. d. Physik., 15, p. 673, 1904.

placement current,

$$V\nabla H = 0 \quad \text{and} \quad S\nabla H = 0. \tag{1}$$

From the second relation we find that H is independent of z, and from the first relation, that it is independent of x and y; hence it is constant inside the coil, and for

$$r \lessgtr r_0 \quad \text{let us assume} \quad H = H_0 \epsilon^{i\omega t}, \tag{2}$$

where ω is $2\pi \times$ frequency, and H_0 the maximum value of H. This of course means an harmonic alternating field. Outside of the coil, H must be constant, but as it vanishes at infinity it is zero everywhere. Hence for

$$r \lessgtr r_m, \quad H = 0. \tag{3}$$

In the material of the coil, Maxwell's equations are

$$4\pi q = V\nabla H, \quad \frac{dH}{dt} = -V\nabla F \quad \text{and} \quad q = \sigma F, \tag{4}$$

where q is the current density,
F is the electric force,
σ is the specific conductivity,

from which follow

$$S\nabla H = 0 \quad \text{and} \quad S\nabla q = 0. \tag{5}$$

Eliminating q and F, by taking the curl of the first equation of (4)

$$4\pi V\nabla q = V\nabla V\nabla H = \nabla S\nabla H - \nabla^2 H$$
$$= -\nabla^2 H \quad \text{by (5)},$$

and from the second and third of (4)

$$V\nabla q = \sigma V\nabla F = -\sigma \frac{dH}{dt},$$

$$\therefore 4\pi\sigma \frac{dH}{dt} = \nabla^2 H. \tag{6}$$

Now assume H in the material of the coil to be given by a function of r alone (on account of symmetry) multiplied by $H_0\epsilon^{i\omega t}$, so that for any point for which $r_0 < r < r_m$,

$$H = H_0 U \epsilon^{i\omega t},$$

where $U = 1$ when $r \leqq r_0$,

$U = 0$ when $r \geqq r_m$,

and $U = f(r)$ for $r_0 < r < r_m$.

In between these points we assume the field to be of the same periodicity but to vary from point to point according to a function of the radius alone. It is also necessary to assume the field continuous as we go from layer to layer.

Equation (6) becomes then in polar coördinates

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} + k^2U = 0, \tag{7}$$

where $k^2 = -i4\pi\sigma\omega$. The solutions of this equation are the Bessel's functions of order zero, J_0 and K_0, and the complete solution is

$$H = H_0\{AJ_0(kr) + BK_0(kr)\}e^{i\omega t}, \tag{8}$$

where A and B are arbitrary constants. So that for

$$\begin{aligned}
r < r_0, \quad & H = H_0 e^{i\omega t}; \\
r_0 < r < r_1, \quad & H_1 = \{A_1 J_0(kr) + B_1 K_0(kr)\} H_0 e^{i\omega t}; \\
r_1 < r < r_2, \quad & H_2 = \{A_2 J_0(kr) + B_2 K_0(kr)\} H_0 e^{i\omega t}; \\
& \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
r_{m-1} < r < r_m, \quad & H_m = \{A_m J_0(kr) + B_m K_0(kr)\} H_0 e^{i\omega t}.
\end{aligned} \tag{9}$$

The $2m$ constants, A_1, B_1, A_2, B_2, $\cdots A_m$, B_m, must be so chosen that, H, the field, is continuous from the inside of the coil to the outside. This will give rise to m equations of condition but which are insufficient to determine the $2m$ constants.

The same current, however, flows in each layer, and the current density is given by the equation

$$4\pi q = \text{curl } H = -\frac{dH}{dr}. \tag{10}$$

So that the total current in any layer per unit length of the current sheet, by integration of (10) over the layer in question is given by

$$\begin{aligned}
4\pi NI &= H_1(r_0) - H_1(r_1) = H_2(r_1) - H_2(r_2) = \text{etc.} \\
&= H_m(r_{m-1}) - H_m(r_m),
\end{aligned} \tag{11}$$

where I is the current, and N the number of turns per unit length axially, of the coil. Equation (11) means that the magnetic field decreases by equal steps as we pass from layer to layer outwards; as the total decrease is from $H_0 e^{i\omega t}$ to 0, and there are m layers, this constant decrease is $H_0 e^{i\omega t}/m$.

So that

$$4\pi NI = \frac{1}{m} H_0 e^{i\omega t},$$

or

$$I = \frac{1}{4\pi m N} H_0 e^{i\omega t}. \tag{12}$$

This, which is equivalent to m equations, see equations (11), gives a sufficient number in conjunction with those of (9) to determine the necessary $2m$ constants.

It may be shown [1] that

$$U = \sqrt{\frac{1}{kr}}\, e^{ikr}$$

and

$$U = \sqrt{\frac{1}{kr}}\, e^{-ikr} \tag{13}$$

are very good approximations to the solutions of the differential equation (7) for large values of kr, and it may also be shown by expanding them that they are valid for *small* values of kr; hence they are very valuable substitutes for the Bessel functions J_0 and K_0. The problem is thus reduced to the domain of elementary functions. The reader is referred for further particulars on this point to Sommerfeld's Papers on Waves on Wires,[2] and on Increase of Resistance with Frequency [3] in the Annalen.

Employing these functions equations (9) give, remembering equation (12):

$$A_1 e^{ikr_0} + B_1 e^{-ikr_0} = \frac{m-0}{m}\sqrt{kr_0},$$

$$A_1 e^{ikr_1} + B_1 e^{-ikr_1} = \frac{m-1}{m}\sqrt{kr_1},$$

. . . . . .

[1] Wied. Ann., 15, loc. cit.
[2] Wied. Ann., 67, p. 233, 1899.
[3] Ann. d. Phys., 15, p. 678.

$$
\begin{aligned}
A_2 e^{ikr_1} + B_2 e^{-ikr_1} &= \frac{m-1}{m}\sqrt{kr_1}, \\
A_2 e^{ikr_2} + B_2 e^{-ikr_2} &= \frac{m-2}{m}\sqrt{kr_2}, \\
&\cdots\cdots\cdots \\
A_m e^{ikr_{m-1}} + B_m e^{-ikr_{m-1}} &= \frac{m-(m-1)}{m}\sqrt{kr_{m-1}}, \\
A_m e^{ikr_m} + B_m e^{-ikr_m} &= \frac{m-m}{m}\sqrt{kr_m}.
\end{aligned} \tag{14}
$$

Consider equations in A_2 and B_2 of equations (14).

Multiply the first by e^{-ikr_2} and the second by e^{-ikr_1} and subtract. This gives

$$
A_2 e^{-ik(r_2-r_1)} + B_2 e^{-ik(r_1+r_2)} = \frac{m-1}{m}\sqrt{kr_1}e^{-ikr_2},
$$

$$
A_2 e^{+ik(r_2-r_1)} + B_2 e^{-ik(r_1+r_2)} = \frac{m-2}{m}\sqrt{kr_2}e^{-ikr_1},
$$

$$
A_2\left(e^{-ik(r_2-r_1)} - e^{ik(r_2-r_1)}\right) = \frac{m-1}{m}\sqrt{kr_1}e^{-ikr_2} - \frac{m-2}{m}\sqrt{kr_2}e^{-ikr_1},
$$

$$
\therefore A_2 = \frac{-\frac{m-1}{m}\sqrt{kr_1}e^{-ikr_2} + \frac{m-2}{m}\sqrt{kr_2}e^{-ikr_1}}{e^{ik(r_2-r_1)} - e^{-ik(r_2-r_1)}}. \tag{15}
$$

In a similar way, multiplying the first equation by e^{ikr_2} and the second by e^{ikr_1} and subtracting we obtain for B_2

$$
B_2 = \frac{\frac{m-1}{m}\sqrt{kr_1}e^{ikr_2} - \frac{m-2}{m}\sqrt{kr_2}e^{ikr_1}}{e^{ik(r_2-r_1)} - e^{-ik(r_2-r_1)}}. \tag{16}
$$

The others are formed in an analogous manner. Hence

$$
H_2 = \left\{A_2\frac{1}{\sqrt{kr}}e^{ikr} + B_2\frac{1}{\sqrt{kr}}e^{-ikr}\right\}H_0 e^{i\omega t},
$$

$$
\begin{aligned}
H_2 = \Bigg\{&\frac{m-1}{m}\sqrt{\frac{r_1}{r}}\,\frac{e^{ik(r_2-r)} - e^{-ik(r_2-r)}}{e^{ik(r_2-r_1)} - e^{-ik(r_2-r_1)}} \\
&- \frac{m-2}{m}\sqrt{\frac{r_2}{r}}\,\frac{e^{ik(r_1-r)} - e^{-ik(r_1-r)}}{e^{ik(r_2-r_1)} - e^{-ik(r_2-r_1)}}\Bigg\}H_0 e^{i\omega t}
\end{aligned} \tag{17}
$$

$$= \left\{ \frac{m-1}{m} \sqrt{\frac{r_1}{r}} \frac{\sin k(r_2 - r)}{\sin k(r_2 - r_1)} - \frac{m-2}{m} \sqrt{\frac{r_2}{r}} \frac{\sin k(r_1 - r)}{\sin k(r_2 - r_1)} \right\} H_0 \epsilon^{i\omega t}. \quad (18)$$

Call

$$r_0 - r = d_0, \qquad r_1 - r = d_1 \cdots \text{etc.}$$

$$r_2 - r_1 = r_3 - r_2 = r_s - r_{s-1} = d,$$

and put $k = h - i$.

Then equation (18) becomes

$$H_2 = \left\{ \frac{m-1}{m} \sqrt{\frac{r_1}{r}} \frac{\sin (d_2 - id_2)h}{\sin (d - id)h} - \frac{m-2}{m} \sqrt{\frac{r_2}{r}} \frac{\sin (d_1 - id_1)h}{\sin (d - id)h} \right\} H_0 \epsilon^{i\omega t}. \quad (19)$$

The conjugate of this obtained by putting $-i$ for i everywhere in equation (19) is

$$\bar{H}_2 = \left\{ \frac{m-1}{m} \sqrt{\frac{r_1}{r}} \frac{\sin (d_2 + id_2)h}{\sin (d + id)h} - \frac{m-2}{m} \sqrt{\frac{r_2}{r}} \frac{\sin (d_1 + id_1)h}{\sin (d + id)h} \right\} H_0 \epsilon^{-i\omega t}, \quad (20)$$

where the dash over a letter denotes the conjugate expression for it.

The electro-magnetic energy of a system is in general:

$$T = \tfrac{1}{2} L I^2,$$

or in this case

$$\tilde{T} = \tfrac{1}{2} L \tilde{I}^2,$$

where

$$\begin{aligned} L &= \text{self-inductance of the system,} \\ I &= \text{current in it,} \\ T &= \text{electro-magnetic energy,} \end{aligned} \quad (21)$$

and where the sign $\sim$ over a variable denotes its mean square value. Whence

$$L = \frac{2\tilde{T}}{\tilde{I}^2} = 2 \frac{\frac{1}{\tau} \int_0^\tau T dt}{\frac{1}{\tau} \int_0^\tau I^2 dt}.$$

To obtain the mean square value of a variable I it is necessary to evaluate the integral

$$\frac{1}{\tau}\int_0^\tau I^2 dt,$$

where τ is the time of a period. For sine or cosine functions we know that $\bar{I}^2 = \frac{1}{2}$ the square of the maximum value of I. But the square of the amplitude of any *imaginary* expression is obtained by multiplying the expression by its conjugate; hence to obtain the mean square value of any function, in imaginaries, of this paper we multiply it by its conjugate and divide it by 2. So that

$$\frac{1}{\tau}\int_0^\tau I^2 dt = \tfrac{1}{2}|I|^2 = \tfrac{1}{2} I\bar{I} = \tfrac{1}{2}\frac{1}{4\pi m N} H_0 e^{i\omega t} \times \frac{1}{4\pi m N} e^{-i\omega t}$$

$$= \tfrac{1}{2}\left(\frac{H_0}{4\pi m N}\right)^2 = \tfrac{1}{2}\frac{H_0^2}{J^2} \quad \text{where} \quad J = 4\pi m N. \tag{22}$$

Hence (21) becomes

$$L = \frac{2\bar{T}}{\frac{1}{2}\frac{H_0^2}{J^2}} = 4\frac{J^2}{H_0^2}\frac{1}{\tau}\int_0^\tau T dt. \tag{23}$$

But

$$T = \frac{1}{8\pi}\int\int\int H^2 dv = \frac{1}{8\pi}\int_0^{2\pi} d\varphi \int H^2 r dr = \tfrac{1}{4}\int_{r=0}^{r=\infty} H^2 r dr \tag{24}$$

per unit length of the coil.

$$\therefore L = \frac{J^2}{H_0^2}\left\{\frac{1}{\tau}\int_0^\tau dt \int_0^\infty H^2 r dr\right\} = \frac{J^2}{H_0^2}\int_0^\infty r dr \frac{1}{\tau}\int_0^\tau H^2 dt$$

$$= \frac{J^2}{2H_0^2}\int_0^\infty H\bar{H} r dr$$

$$= \frac{J^2}{2H_0^2}\left\{\int_0^{r_0} H\bar{H} r dr + \int_{r_0}^{r_1} H\bar{H} r dr + \int_{r_1}^{r_2} H\bar{H} r dr + \cdots + \int_{r_{m-1}}^{r_m} H\bar{H} r dr\right\}, \tag{25}$$

$$= \frac{J^2}{2H_0^2}\{T_0 + T_1 + T_2 + \cdots + T_m\}, \tag{26}$$

$$T_0 = \int_0^{r_0} r dr H_0 e^{i\omega t} H_0 e^{-i\omega t} = \int_0^{r_0} r dr H_0^2 = \frac{H_0^2}{2} r_0^2. \tag{27}$$

Let us find an expression for $H_2\bar{H}_2$ by multiplying together equations (19) and (20).

$$H_2\bar{H}_2 = \left[\left(\frac{m-1}{m}\right)^2 \frac{r_1}{r} \frac{\sin(hd_2 - ihd_2)\sin(hd_2 + ihd_2)}{\sin(hd - ihd)\sin(hd + ihd)} + \left(\frac{m-2}{m}\right)^2 \frac{r_2}{r}\right.$$

$$\times \frac{\sin(hd_1 - ihd_1)\sin(hd_1 + ihd_1)}{\sin(hd - ihd)\sin(hd + ihd)} - \left(\frac{m-1}{m}\right)\left(\frac{m-2}{m}\right)\sqrt{\frac{r_1 r_2}{rr}} \qquad (28)$$

$$\left.\times \frac{\sin(hd_1 + ihd_1)\sin(hd_2 - ihd_2) + \sin(hd_1 - ihd_1)\sin(hd_2 + ihd_2)}{\sin(hd - ihd)\sin(hd + ihd)}\right] \times H_0^2.$$

Expression (28) may be reduced by means of the following formulæ:

$$\sin(a_1 + ia_1)\sin(a_2 - ia_2) + \sin(a_1 - ia_1)\sin(a_2 + ia_2)$$

$$= \cos(a_1 - a_2)\cos i(a_1 + a_2) - \cos(a_1 + a_2)\cos i(a_1 - a_2) \qquad (29)$$

$$= \cos(a_1 - a_2)\cosh(a_1 + a_2) - \cos(a_1 + a_2)\cosh(a_1 - a_2)$$

and

$$\sin(a - ia)\sin(a + ia) = \tfrac{1}{2}\cos 2ia - \tfrac{1}{2}\cos 2a$$

$$= \frac{\cosh 2a - \cos 2a}{2} \qquad (30)$$

to

$$H_2\bar{H}_2 = \left[\left(\frac{m-1}{m}\right)^2 \frac{r_1}{r} \frac{\cosh 2hd_2 - \cos 2hd_2}{\cosh 2hd - \cos 2hd} + \left(\frac{m+1}{m}\right)^2 \frac{r_2}{r}\right.$$

$$\times \frac{\cosh 2hd_1 - \cosh 2hd_1}{\cosh 2hd - \cos 2hd} - 2\left(\frac{m-1}{m}\right)\left(\frac{m-2}{m}\right)\sqrt{\frac{r_1 r_2}{rr}} \qquad (31)$$

$$\left.\times \frac{\cosh(d_2 + d_1)h\cos(d_2 - d_1)h - \cosh(d_2 - d_1)h\cos(d_2 + d_1)h}{\cosh 2hd - \cos 2hd}\right] H_0^2.$$

This transformation may also be accomplished by substituting $h - i$ for k in equation (17), forming the conjugate, multiplying out, and remembering that

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

and that

$$\cosh y = \frac{e^{y} + e^{-y}}{2}, \text{ etc.}$$

It is now necessary to integrate this expression multiplied by rdr

from r_1 to r_2. For example

$$\int_{r_1}^{r_2} \frac{1}{r} \cosh 2h(r_2 - r) r dr = -\frac{1}{2h} \int_{r_1}^{r_2} \cosh 2h(r_2 - r)(-2h dr)$$

$$= -\frac{1}{2h}\left[\sinh 2h(r_2 - r_2) - \sinh 2h(r_2 - r_1)\right] = \frac{1}{2h} \sinh 2hd. \quad (32)$$

And

$$\int_{r_1}^{r_2} \tfrac{1}{2} \cosh 2h\left(\frac{r_1 + r_2}{2} - r\right) r dr$$

$$= -\frac{1}{2h} \int_{r_1}^{r_2} \cosh 2h\left(\frac{r_1 + r_2}{2} - r\right)(-2h dr)$$

$$= -\frac{1}{2h}\left[\sinh 2h\left(\frac{r_1 + r_2}{2} - r_2\right) - \sinh 2h\left(\frac{r_1 + r_2}{2} - r_1\right)\right]$$

$$= -\frac{1}{2h}\left[\sinh h(r_1 - r_2) - \sinh h(r_2 - r_1)\right]$$

$$= \frac{1}{h} \sinh h(r_2 - r_1) = \frac{1}{h} \sinh hd, \text{ etc.} \quad (33)$$

Carrying out these integrations and putting $x = 2hd$ the integrated expression simplifies into

$$\int_{r_1}^{r_2} H_2 \bar{H}_2 r dr = \left[\left\{\left(\frac{m-1}{m}\right)^2 \frac{r_1 d}{3} + \left(\frac{m-2}{m}\right) \frac{r_2 d}{3}\right\}\left\{\frac{3}{x} \frac{\sinh x - \sin x}{\cosh x - \cos x}\right\}\right. \quad (34)$$

$$\left. + \left\{\left(\frac{m-1}{m}\right)\left(\frac{m-2}{m}\right) \frac{d\sqrt{r_1 r_2}}{3}\right\}\left\{\frac{12}{x} \frac{\cosh \tfrac{1}{2}x \sin \tfrac{1}{2}x - \sinh \tfrac{1}{2}x \cos \tfrac{1}{2}x}{\cosh \tfrac{1}{2}x \cosh \tfrac{1}{2}x - \cos \tfrac{1}{2}x \cos \tfrac{1}{2}x}\right\}\right] H_0^2.$$

Remembering equation (25) and substituting in it equations (27) and (34) we obtain

$$L_\omega = \frac{J^2}{4}\left\{r_0^2 + \tfrac{2}{3} \frac{d}{m^2}(Q'P' + Q''P'')\right\}, \quad (35)$$

where

$$Q' = \frac{3}{x} \frac{\sinh x - \sin x}{\cosh x - \cos x} \quad \text{and} \quad \begin{matrix} Q' = 1 \text{ when } x = 0, \\ Q' = 0 \text{ when } x = \infty, \end{matrix} \quad (36)$$

where

$$Q'' = \frac{12}{x}\frac{\cosh x/2 \sin x/2 - \sinh x/2 \cos x/2}{\cosh x/2 \cosh x/2 - \cos x/2 \cos x/2} \tag{37}$$

and $Q'' = 1$ when $x = 0$, $Q'' = 0$ when $x = \infty$,

where

$$P' = m^2 r_0 + 2(m-1)^2 r_1 + 2(m-2)^2 r_2 + 2(m-3)^2 r_3 + 2r_{m-1} \tag{38}$$

and where

$$\begin{aligned} P'' = {} & m(m-1)\sqrt{r_0 r_1} + (m-1)(m-2)\sqrt{r_1 r_2} \\ & + [m-(m-2)][m-(m-1)]\sqrt{r_{m-2} r_{m-1}}. \end{aligned} \tag{39}$$

Since for $x = \infty$ equation (36) becomes

$$L_\infty = \frac{J^2}{4} r_0^2 = 4\pi^2 m^2 N^2 r_0^2 \tag{40}$$

we may write it

$$\frac{L_\omega - L_\infty}{L_\infty} = \frac{2}{3}\frac{d}{r m_0^2}(Q'P' + Q''P'') \tag{41}$$

Equation (40) gives the curious result that for infinite frequency the self-inductance of a coil of m-layers, and consequntly having md units of thickness of winding, diminishes to what it would be were the total current all flowing at distance r. That is, it diminishes in *absolute* amount as much as if it were a single layer, md units thick. But of course the self-inductance of m-layers being approximately m^2 times that of a single layer, the total *fractional* change is m times less in amount.

This result may be explained by a consideration of the following equations taken from Sommerfeld's[1] paper and previously explained by Wien,[2] which give the value of the current density at the bounding surfaces of the layers, where for infinite frequency it is mostly concentrated.

$$\begin{aligned} &\text{For } r = r_0, && 4\pi q_1 = ikH_0 e^{i\omega t}; \\ &r = r_1, && -4\pi q_1 = 4\pi q_2 = ik\frac{m-1}{m} H_0 e^{i\omega t}; \\ &r = r_2, && -4\pi q_2 = 4\pi q_3 = ik\frac{m-2}{m} H_0 e^{i\omega t}. \end{aligned} \tag{42}$$

[1] Ann. d. Phys., 15; loc. cit., p. 698.
[2] Ann. d. Phys., 14; loc. cit., p. 9.

It is here seen that the current decreases $1/m^{\text{th}}$ in amplitude as we go from one layer to the next, and that at the bounding surfaces of the layers the currents are *equal* in amount, but *opposite* in phase, so that in the material of the wire there is current, but no magnetic field.

For $x = 0$ (41) becomes

$$\frac{L_0 - L_\infty}{L_\infty} = \tfrac{2}{3}\frac{d}{r_0^2 m^2}(P' + P''). \tag{43}$$

This is then the total possible fractional change.

If $m = 1$, $P' = r_0$ and $P'' = 0$, (42) becoming

$$\frac{L_\omega - L_\infty}{L_\infty} = \tfrac{2}{3}\frac{d}{r_0} Q'$$

for a coil of a single layer. Dividing (41) by (43) we obtain

$$\frac{L_\omega - L_\infty}{L_0 - L_\infty} = \frac{Q'P' + Q''P''}{P' + P''} \tag{44}$$

which represents the fraction of the total possible change still to take place at frequency ω where $\omega = 2\pi n$ and $n =$ number of alternations per second,

$$= \tfrac{1}{2}(Q' + Q'') + \tfrac{1}{2}\frac{P' - P''}{P' + P''}(Q' - Q'')$$

$$= \tfrac{1}{2}(Q' + Q'') + P\tfrac{1}{2}(Q' - Q'') \tag{45}$$

where

$$P = \frac{P' - P''}{P' + P''}$$

and $P = 1$ when $m = 1$.

To facilitate computation the two functions Q' and Q'' have been carefully calculated and plotted for different values of

$$x = 2hd = 2d\sqrt{2\pi\omega\sigma} = (4\pi d\sqrt{\sigma})\sqrt{n}. \tag{46}$$

In Table I. are the data used in the computation.

The mathematical tables of Vladimir Vassal were employed in finding the values of cosh x and of sinh x for different values of x.

Equations (38) and (39) give for different values of m, remembering that $r_n = r_0 + nd$, where d is the thickness of the wire.

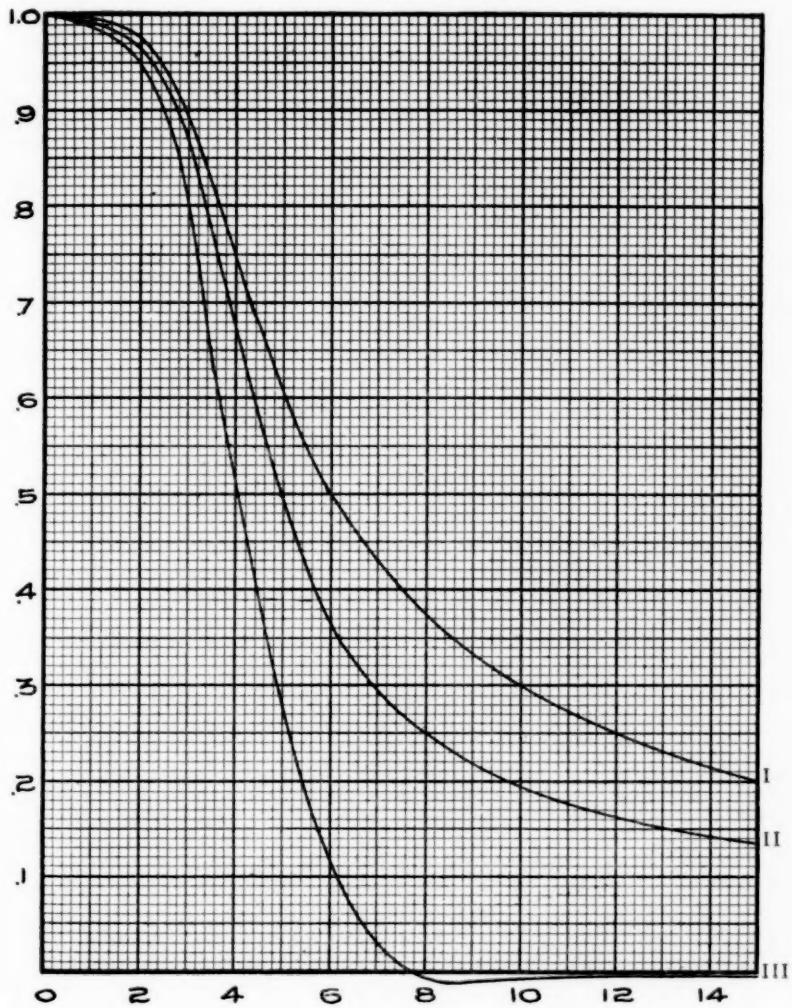

Fig. 2.

I (upper curve) is Q'; II (middle curve) is $Q' - \frac{1}{3}(Q' - Q'')$; III (lower curve) is Q''.

And assuming in (39) that $\sqrt{r_{m-1} r_m} = \frac{1}{2}(r_{m-1} + r_m)$, which is always very approximately true, as r_{m-1} and r_m have but slightly different values.

TABLE I.

Values for the Computation of the Functions Q′ and Q″.

x	cosh x	sinh x	cos x	sin x	θ	Q'	Q''
0.00	1.0000	0.0000	1.0000	0.0000	0.00	1.0000	1.000
0.25	1.0314	0.2526	0.9689	0.2473	14.32	——	
0.50	1.1276	0.5211	0.8776	0.4789	28.65	0.9999	0.993
1.00	1.5431	1.1752	0.5402	0.8415	57.30	0.9984	0.992
1.50	2.3523	2.1291	0.0706	0.9975	——	0.9920	
1.75	2.964	2.790	−0.1788	0.9839	100.3	——	
2.00	3.7622	3.6269	−0.4163	0.9092	——	0.9756	0.951
2.50	6.133	6.051	−0.8018	0.5990	143.2	——	
3.00	10.0677	10.0178	−0.9900	0.1490	——	0.8932	
3.50	16.578	16.547	−0.9348	−0.3551	171.9	——	0.793
3.60	18.30	18.28	——	——	——	——	0.667
3.65	19.244	19.218	−0.8738	−0.4863	209.1	——	
3.80	22.33	22.31	−0.7902	−0.6129	——	——	
3.90	24.72	24.70	−0.7218	−0.6921	217.8	——	
4.00	27.308	27.262	−0.6534	−0.7570	——	0.7515	0.529
4.50	45.23	45.22	−0.2096	−0.9778	257.9	——	
5.00	74.204	74.197	+0.2840	−0.9588	——	0.6100	0.276
6.00	201.8	201.8	0.9537	−0.3007	343.5	0.5032	0.1202
7.00	547.8	547.8	0.7537	+0.6574	——	0.4287	0.0299
7.30	740	740	——	——	——	——	0.0165
7.60	997	997	——	——	——	——	0.00631
7.80	1,224	1,224	——	——	——	——	0.00088
8.00	1,490	1,490	−0.1461	+0.989	458.4	0.3750	−0.0029
9.00	4,080	4,080	——	——	——	0.3333	−0.0113
10.00	11,000	11,000	−0.839	−0.545	573	0.3000	−0.010
12.00	81,400	81,400	——	——	——	0.2500	−0.0062
16.00	4,440,000	4,440,000	−0.9563	−0.293	917	0.1875	−0.0003
∞	∞	∞	Indet.	Indet.	∞	0.0000	0.0

m	P'	P''	m	P'	P''
1	r_0	0	5	$85r_0 + 100d$	$40r_0 + 50d$
2	$6r_0 + 2d$	$2r_0 + d$	6	$146r_0 + 210d$	$70r_0 + 105d$
3	$19r_0 + 12d$	$8r_0 + 6d$	etc.	.....	.....
4	$44r_0 + 40d$	$20r_0 + 20d$			

One easily notices that

$$P' = 2P'' + mr_0. \tag{47}$$

So that

$$P = \frac{P' - P''}{P' + P''} = \frac{P'' + mr_0}{3P'' + mr_0}. \tag{48}$$

It may be shown by the method of differences that the coefficients of r_0 and d in P' are given in general by

$$\rho_m = 2(m-1) + 2(m-1)(m-2) + \frac{(m-1)(m-2)(m-3)}{3} \quad (49)$$

and

$$\delta_m = (m-1) + 2(m-1)(m-2) + \tfrac{5}{6}(m-1)(m-2)(m-3) + \tfrac{1}{12}(m-1)(m-2)(m-3)(m-4). \quad (50)$$

So that

$$P = \tfrac{1}{3}\frac{\left(\rho_m + \delta_m \frac{d}{r_0}\right) + m}{\left(\rho_m + \delta_m \frac{d}{r_0}\right) + m}. \quad (51)$$

For $m = 1$ this expression is 1 and for m large it approaches $\frac{1}{3}$, so that the function of equation (44) may be written more conveniently

$$\frac{L_\omega - L_\infty}{L_0 - L_\infty} = Q' - (1-P)\tfrac{1}{2}(Q' - Q''), \quad (52)$$

which gives the law according to which the fractional change in inductance takes place, and which varies for $m = 1$ and m large from Q' to $Q' - \frac{1}{3}(Q' - Q'')$.

The curves Q' and $Q' - \frac{1}{3}(Q' - Q'')$ are inserted in the plot. Hence all the curves which show the law of falling off of self-inductance in coils of any number of layers, lie in the region between Curves I. and II., and as is easily seen, are of nearly the same shape.

Having thus the two curves Q' and Q'' it is easy to construct the characteristic curve for the coil in question and then to derive a calibration table from it.

To obtain the characteristic curve it is necessary to calculate the value of P for a given m, r and d. Having done this subtract from the ordinates of Q', $(1 - P)$ times the vertical distance between the curves Q' and Q''; this amounts to dividing the vertical space between the two curves in the ratio $1 - P$ to 1. The curve drawn through the points, thus derived, will be the curve showing the way in which the self-inductance of that particular coil falls off.

From this curve it is now necessary to derive a second one, by

the following process: $x = \{4\pi d\sqrt{\sigma}\}\sqrt{n}$ is the relation between x and n for a coil wound with wire of conductivity σ, and of diameter d, so that every value of x corresponds to a definite frequency. By assuming values of n differing by equal intervals and plotting against these the values of the ordinates above the x's corresponding to the n's, a correction curve is derived for the coil for any frequency.

APPROXIMATIONS FOR LOW AND HIGH FREQUENCIES.

For very small frequencies an expression for (52) much more convenient for calculation may be derived by expanding the cosh $x/2$, sinh $x/2$, cos $x/2$, sin $x/2$, cosh x, and cos x by means of their expansions:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots; \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots;$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots; \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots;$$

and neglecting powers of x higher than the fourth. It was found convenient to employ the formula:

$$\cosh\frac{x}{2}\sin\frac{x}{2} - \sinh\frac{x}{2}\cos\frac{x}{2} = \frac{1+i}{2}\sin\frac{1+i}{2}x + \frac{1-i}{2}\sin\frac{1-i}{2}x$$

$$= \frac{a}{2}\sin\frac{ax}{2} + \frac{b}{2}\sin\frac{bx}{2}, \text{ say.}$$

$$= \frac{a}{2}\left(\frac{ax}{2} - \frac{a^3x^3}{3!\,2^3} + \frac{a^5x^5}{5!\,2^5} - \cdots\right) + \frac{b}{2}\left(\frac{bx}{2} - \frac{b^3x^3}{3!\,2^3} + \frac{b^5x^5}{5!\,2^5} - \cdots\right)$$

$$= x\frac{(a^2+b^2)}{2^2} - \frac{x^3}{3!}\left(\frac{a^4+b^4}{2^4}\right) + \frac{x^5}{5!}\left(\frac{a^6+b^6}{2^6}\right) - \frac{x^7}{7!}\left(\frac{a^8+b^8}{2^8}\right) + \cdots.$$

But since

$$a = 1+i, \quad a^2 = 2i, \qquad a^4 = -4, \quad a^6 = -8i, \quad a^8 = 16$$

$$b = 1-i, \quad b^2 = -2i, \quad b^4 = -4, \quad b^6 = +8i, \quad b^8 = 16,$$

this becomes:

$$= \frac{x^3}{3!}\frac{8}{2^4} - \frac{x^7}{7!}\frac{32}{2^8},$$

$$\cosh x - \cos x = 2\left(\frac{x^2}{3!} + \frac{x^6}{6!} + \frac{x^{10}}{10!}\right),$$

$$Q'' = 6\frac{\left(\frac{1}{3!}\frac{8}{2^4} - \frac{x^4}{7!}\frac{32}{2^8}\right)}{\left(\frac{1}{2!} + \frac{x^4}{6!}\right)} = \frac{1 - \frac{3}{2}\frac{1}{7!}x^4}{1 + \frac{2x^4}{6!}} = \left(1 - \frac{31}{2}\frac{x^4}{7!}\right),$$

$$Q' = 3\frac{\left(\frac{1}{3!} + \frac{x^4}{7!}\right)}{\left(\frac{1}{2!} + \frac{x^4}{6!}\right)} = \frac{1 + \frac{6x^4}{7!}}{1 + \frac{2x^4}{6!}} = \left(1 - \frac{8x^4}{7!}\right),$$

$$Q' - Q'' = \frac{\frac{15}{2}\frac{x^4}{7!}}{1 + \frac{2x^4}{6!}}.$$

By means of these approximations equation (52) may be written

$$\frac{L_\omega - L_\infty}{L_0 - L_\infty} = 1 - \frac{8x^4}{7!}[1 + (1 - P)]. \qquad (53)$$

$(1 - P)/2$ varies from zero to $\frac{1}{3}$, so that the two limiting approximation curves are $1 - 8x^4/7!$ and $1 - \frac{4}{3}\,8x^4 7!$, for small values of x. $(x < 2.)$

For high frequencies, which correspond to large values of x, we may put $Q'' = 0$; as a consideration of the function shows that it is practically zero for $x > 8$.

Doing this, equation (52) becomes

$$\frac{L_\omega - L_\infty}{L_0 - L_\infty} = \frac{3}{2x}(1 + P). \qquad (54)$$

For a coil of a single layer, for which $P = 1$, this becomes $3/x$, so Curve I. beyond abscissæ greater than $x = 7$ is *very* approximately a hyperbola. It is easily seen that all the intermediate curves are also hyperbolæ for $x > 7$.

Summary.

Following a method analogous to that of A. Sommerfeld, an expression is found giving the change of self-inductance with the frequency, of a long cylindrical coil of any number of layers.

Calling λ the fraction of the total possible change, still unchanged at frequency $m = 2\pi n$, where $n =$ number of alternations per sec., the expression obtained may be written

$$\lambda = \frac{L_\omega - L_\infty}{L_0 - L_\infty} = Q' - \tfrac{1}{2}(1 - P)(Q' - Q''),$$

where

$$Q' = \frac{3}{x}\frac{\sinh x - \sin x}{\cosh x - \cos x} \quad \text{and} \quad \begin{matrix} Q' = 1 \text{ when } x = 0 \\ Q' = 0 \text{ when } x = \infty \end{matrix},$$

where

$$Q'' = \frac{12}{x}\frac{\cosh x/2 \sin x/2 - \sinh x/2 \cos x/2}{\cosh x/2 \cosh x/2 - \cos x/2 \cos x/2} \text{ and } \begin{matrix} Q'' = 1 \text{ when } x = 0 \\ Q'' = 0 \text{ when } x = \infty \end{matrix},$$

and where P is a function of the dimensions of the coil alone, and which varies from 1 to $\frac{1}{8}$ for $m = 1$ and m large, respectively. See equation 51.

Since a coil of a single layer is of particular interest, the formulæ for this case, which are much less complex than the above, will be given.

For $m = 1$, $P = 1$.

Then

$$\lambda = \frac{3}{x}\frac{\sinh x - \sin x}{\cosh x - \cos x}.$$

For small values of x ($x < 2$) this may be written

$$\lambda = 1 - \frac{8x^4}{7!}$$

and for large values of x ($x > 7$)

$$\lambda = \frac{3}{x}.$$

These values agree perfectly with the results, deduced solely for a coil of a single layer, published in the Bulletin of the Bureau of Standards and in the Proc. A. A. A. S., Vol. 41, No. 34, 1906.

Since $4\pi^2 N^2 r_0^2$ is the self-inductance of a coil of a single layer of mean diameter r_0, the calculations show (see equation 40) that with

increasing frequency, the mean radius of a coil of a single layer decreases by the radius of the wire, with which it is wound. This gives a means of calculating the total possible change in self-inductance.

Equation 40 also shows that the mean radius of a coil of m layers, decreases to the minimum radius r_0 of the coil, just as if it were of a single layer wound with wire md units thick. The fractional change in self-inductance, however, is far less than for a coil of one layer, as the self-inductance is approximately m^2 greater.

By means of a plot of the functions Q' and Q'', the characteristic curve for any given cylindrical coil may be readily calculated and a table of corrections derived for the self inductance at any frequency.

CLARK UNIVERSITY,
March, 1906.

ON THE VELOCITY OF SOUND IN GASES, AND THE RATIO OF THE SPECIFIC HEATS, AT THE TEMPERATURE OF LIQUID AIR.

BY S. R. COOK.

Historical. — The determination of the velocity of sound, at temperatures lower than that obtainable in free space, has been made possible within the past few years by the production, in sufficient quantities, of liquid air and other refrigerating fluids. When the present work was planned (summer of 1903) no measurements had been published on the velocity of sound at the temperature of liquid air; and it was not until the work was well under way that my attention was called to a paper on the velocity of sound in nitrogen, which had just appeared in the "Annalen der Physik."[1] The author had used a method similar to that which I had planned; but, as so little work had been done in this particular field, it was considered of sufficient importance to continue the investigation, using gases other than nitrogen.

Greely,[2] during his expedition to the far north, made a great number of determinations of the velocity of sound in free air, by the eye-and-ear-method, over accurately measured courses. The determinations were made attemperatures ranging from 0° F. to −79° F., and it was found that within this range the decrease in the velocity of sound was 60.3 cm. per degree while the ratio of the two specific heats was constant within the accuracy of the measurements.

Ratio of the Specific Heats. — A. W. Witkowski[3] and A. Léduc[4] have recently been investigating the specific heats of gases, and have found that there appears to be a variation with both pressure and temperature. The variation with temperature seems very slight,

[1] S. Valentiner, Ann. der Physik, Drude 15, pp. 74–106, 1904.
[2] Greely, Meterolog. Zeitschrift, VII., p. 6, 1890; Phil. Mag. (5), 30, 1890.
[3] Witkowski, Acad. Sci. Cracovie Bull., 16, pp. 138–157, 1899.
[4] Léduc, Annal. Chim. Phys., [7] 17, pp. 484–510, 1899.

while that with pressure is as great as 10 per cent. at 100 atmospheres. E. H. Stevens,[1] by means of a heated resonance tube, has measured the velocity of sound at temperatures 950° and 1000°, and finds the ratio of the specific heats to be 1.34 for air; while A. Kalahne,[2] by an acoustic method, studied the variation of the ratio of the specific heats from 0° to 900° and finds that the ratio of the specific heats does not vary up to 450°. At 700° it has fallen four parts in one thousand, and at 900° the ratio is 1.39 for air.

From the variation of these experimental results, it would seem that the data were still lacking for the establishment of the dependence of the ratio of the specific heats upon temperature. The results obtained for the dependence of the ratio of the specific heats upon pressure show, in general, a decrease both for increase and decrease of pressure, with the greatest variation at the temperature − 136°.

L. Lohncke[3] has studied the variation of the specific heats of the vapor of mercury, and finds no variation with temperature. As the ratio of the specific heats is indirectly a measure of the ratio of translational to the internal energies of the molecules of the gas, and as the vapor of mercury is known to be monatomic, this result might have been expected. It is not so easy, however, to predict the division of energy in a diatomic or polyatomic molecule, with wide ranges of temperature and pressure conditions. Capstick[4] has shown that for a number of organic compounds the ratio of the specific heats of the vapors and therefore the ratio of the translational to the internal energy of molecules of polyatomic vapors, depends on the number of atoms in the molecule, and is independent of the specific gravity of the vapor.

It was thought that, if there were any variation in the ratio of the specific heats, and therefore in the division of energy between the translational and intermolecular energies of the molecules of the diatomic gases, it would be most likely to occur when these gases were in their vapor condition, or very close to the point of condensation. It was determined, therefore, to study atmosphere air and

[1] Stevens, Deutsche Phys. Gesell. Verh., 3, 5, pp. 54–56, 1901.
[2] Kalahne, Ann. d. Physik., 112, pp. 225–256, 1903.
[3] Lohncke, Wied. Ann., 66 (1), pp. 111–115, 1898.
[4] Capstick, Phil. Trans., 185A, p. 1, 1894; 186A, p, 567, 1895.

oxygen. The critical pressure for air being somewhat greater, and that for oxygen somewhat less than atmospheric pressure at the temperature of liquid air.

From the well known work of Regnault, who determined the specific heat at constant pressure of air, hydrogen, carbon dioxide, and other gases and vapors, from 0° to 200° at pressures between 1 and 12 atmospheres, it has been concluded that for the so-called permanent gases the specific heats do not vary with temperature or pressure, but for the more compressible gases and vapors a small variation is quite probable.

For an ideal gas, in which the influence of the forces due to the mass of the molecules may be neglected in comparison with the kinetic energy of the free molecule, and in which the volume of the molecular mass may be neglected in comparison with the whole volume of the gas — *i. e.*, a gas in which the Boyle-Gay-Lussac-Avogadro Law maintains, the volume may be considered as a simple function of the temperature and pressure; so we may write the relation:

$$dv = \left(\frac{\partial v}{\partial p}\right)_\theta dp + \left(\frac{\partial v}{\partial \theta}\right)_p d\theta, \tag{1}$$

where v, p, and θ are the volume, pressure, and absolute temperature respectively, and the subscripts indicate the variables that have been regarded as constant during the respective variations. We may also write the fundamental relation:

$$dp = \left(\frac{\partial p}{\partial v}\right)_\theta dv + \left(\frac{\partial p}{\partial \theta}\right)_v d\theta, \tag{2}$$

in which the symbols have the same meaning as in equation (1). Either one of these fundamental equations is sufficient to determine the thermodynamic conditions of an ideal gas.

For an ideal gas we have also the characteristic equation:

$$pv = R\theta. \tag{3}$$

It may be shown[1] that if C_v and C_p are the specific heats, respectively,

[1] Planck, Thermodynamics, p. 56.

$$C_v = \left(\frac{\partial u}{\partial \theta}\right)_v, \tag{4}$$

and

$$C_p = \left\{\left(\frac{\partial u}{\partial v}\right)_p + p\right\}\left(\frac{\partial v}{\partial \theta}\right)_p. \tag{5}$$

Since the velocity of sound through a gas depends on the ratio of its specific heats, this ratio may be conveniently determined by measuring the velocity of sound.

For the velocity of sound, we have the equation,

$$u^2 = -\gamma v^2 \left(\frac{\partial p}{\partial v}\right)_\theta. \tag{6}$$

where u is the velocity of sound and γ the ratio of the specific heats.

Equation (6) may be written

$$u^2 = \gamma p v \left\{1 - \frac{1}{p}\left(\frac{\partial (pv)}{\partial v}\right)_\theta\right\}. \tag{7}$$

Hence we may write

$$\gamma = \gamma' \rho \left(\frac{\lambda}{\lambda'}\right)^2 \left\{1 - \frac{1}{p}\left(\frac{\partial (pv)}{\partial v}\right)\right\}, \tag{8}$$

where γ and γ', and λ' and λ' are the ratios of the specific heats, and the wave-lengths of sound, respectively.

The ratio of the specific heats for any gas is determined by the ratio of the internal, or atomic, energy of the molecule to the transitional energy of the molecule. If, then, a certain quantity of energy U be given to a molecule by the application of heat, and if U_1 is the part of U taken up by the atoms of the molecule, and U_2 the part taken by the molecule to increase its kinectic energy of translation, and if β is the ratio of the atomic to the molecular energies: then

$$\beta = \frac{U_1}{U_2} \tag{9}$$

and for a perfect gas we have the relation,

$$\beta + 1 = \frac{2}{3(\gamma - 1)}. \tag{10}$$

Equation (10) maintains for an ideal gas in which equation (3) is the characteristic equation. For a vapor or gas which does not follow the Boyle-Gay-Lussac-Avogadro law, we may write for the characteristic equation

$$pv = R\theta + \varphi(p, v, \theta).$$

From equations (3), (4), (5), (10) and (11), we may easily deduce the functional equation:

$$\gamma - 1 = \frac{2}{3(\beta + 1)} \left\{ 1 + \frac{1}{v}\frac{\partial \varphi}{\partial p} + \frac{1}{p}\frac{\partial p}{\partial v} + \frac{2}{R}\frac{\partial \varphi}{\partial \theta} + \cdots - \frac{\varphi}{pv} - \frac{\varphi}{pv^2}\frac{\partial \varphi}{\partial p} - \frac{\varphi}{p^2 v}\frac{\partial \varphi}{\partial v} - \frac{2\varphi}{Rpv}\frac{\partial \varphi}{\partial \theta} \cdots \right\}. \tag{12}$$

Writing the characteristic equation of Clausius in the functional form:

$$R\theta = p(v - a) + \frac{K}{\theta(v + b)^2}, \tag{13}$$

and making the assumption that a and b, for air at normal temperature and pressure, are small in comparison with v; and neglecting quantities of the second order, we find,

$$\beta + 1 = \tfrac{2}{3}\frac{1}{(\gamma - 1)} \left\{ 1 + \frac{4}{p}\frac{\partial (pv)}{\partial v} \right\}, \tag{14}$$

writing Van der Waals's equation in the functional form

$$pv = R\theta + \left(\frac{ab}{v^2} + pb - \frac{a}{v} \right), \tag{15}$$

and making similar assumptions we have,

$$\beta + 1 = \tfrac{2}{3}\frac{1}{\gamma - 1} \left\{ 1 + \frac{2}{p}\frac{\partial (pv)}{\partial v} \right\}, \tag{16}$$

APPARATUS FOR DETERMINING THE VELOCITY OF SOUND.

The apparatus for determining the velocity of sound in gases or vapors at low temperature was the usual form of the double Kundt sound apparatus. A glass vibrating rod, one meter in length and 1.14 cm. in diameter was clamped for its first overtone. Sound tubes, of similar dimensions, were fitted to each node, by suitable

stoppers, with delivery tubes for the introduction of dry air and the gases to be experimented with. One of these sound tubes was fitted in a long, cylindrical, unsilvered Dewar bulb made especially for these experiments. This bulb was 72 cm. long and 7.8 cm. external diameter; and was fitted to the sound tube by two rubber stoppers, carefully ground in shape and forced in by wooden clamps, clamped together by tension screws. The delivery tubes, for the entrance of the liquid air and the escape of the vapor, passed through these stoppers. It was found necessary to have two stoppers, because the inner stopper, which was in contact with the liquid air, when the bulb was filled, became hard and contracted away from the Dewar bulb; so that the Dewar bulb could not be made to contain the liquid air. It was also found, that if the stoppers were cut so that when they were forced over the sound tube they were under tension transversely, the extra tension due to the contraction at liquid air temperature caused the stoppers to split longitudinally at the point where they were pierced by the delivery tubes. In order to avoid this, the stoppers were cut in such a manner that when clamped in position by the screw clamps, they were under compression. With this arrangement it was found possible to fill the Dewar bulb with liquid air and maintain the temperature quite constant for an hour or more.

In order to measure the temperature at which the velocity of sound was determined, a resistance thermometer was introduced into the sound tube. This was supported by a wire frame which fitted into the tube. The wires for the thermometer and the leads were led through the stopper, which closed one end of the sound tube; the other end being sealed. The device used to adjust the length of the air column in the sound tube consisted of a cork fitted to the end of a long glass rod (which was also passed through the stopper). In order to separate this part of the apparatus completely from the standard sound tube, in which air was always the medium, a stopper was fitted inside the vibrating tube A, at one of its nodes. Corks were also fitted to the ends of the vibrating tube by means of fish glue. The fish glue was found to stand the temperature of liquid air better than other forms of glue, although even it became quite hard and brittle at low temperature. The standard tube was fitted

with a cork, which could be adjusted by means of a glass tube, passing through the stopper at the farther end of the tube. This part of the apparatus is shown in diagrammatic form in Fig. 1.

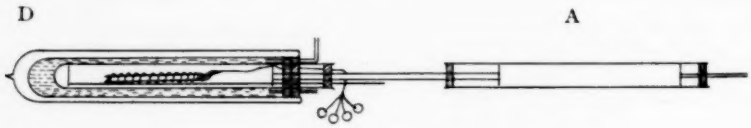

Fig. 1.

The pressure in the sound tube could be measured by an open mercury manometer.

To Fill the Dewar Bulb. — When the apparatus was all in place, it was necessary to fill the Dewar bulb with liquid air, while it remained in a horizontal position, in order that the cork filings, used to mark the position of the nodes, should not be disturbed. In order to lead in the liquid air against the pressure of the air evaporating in the large Dewar bulb, a third cylindrical Dewar bulb, unsilvered, was fitted with three tubes entering a rubber stopper, clamped into place by clamps similar to those on the Dewar bulb *D*. One of these tubes reached nearly to the bottom, the other two passing only through the stopper. The first of these tubes was connected to the delivery tube leading into the Dewar bulb *D*. One of those, that passed only through the stopper, was connected to the compression siphon, which in turn was fitted to the spherical retaining Dewar bulbs; the other was left open to the air. When this small cylindrical bulb had been filled with the liquid air, the rubber tube connecting it to the spherical Dewar bulb was closed by a pinch cock, the one that had been open to the air was also closed, and the liquid air was forced from this bulb into the horizontal cylindrical bulb *D*, by the pressure produced by the vaporization of the air in the small cylindrical Dewar bulb. The apparatus and connections are shown in Fig. 2.

Measurements of Temperature. — The temperature of the standard sound tube was determined by two mercury thermometers reading to tenths of a degree centigrade, suspended with their bulbs in contact with the outer surface of the glass tube.

The temperature in the comparison sound tube was measured by

a resistance thermometer, mounted as near as possible to the center of the portion of the tube in which the sound waves were produced. A copper thermometer, instead of the usual platinum thermometer, was used. The copper thermometer had the advantage of having no junctions of dissimilar metals at points of varying temperature, which condition could not have been avoided in a platinum thermometer, except by leading the platinum wires outside the Dewar bulb.

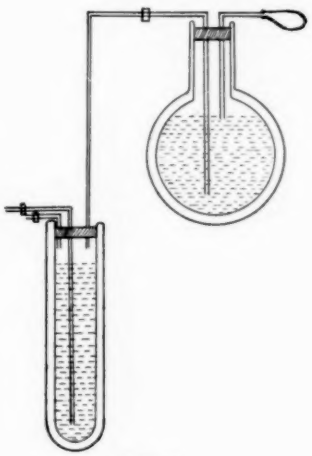

Fig. 2.

The Copper Thermometer. — The copper thermometer was made by winding, noninductively, about eight meters of double silk insulated copper wire, diameter .018 cm., on a small glass tube 17.7 cm. in length. The ends of these wires were led out through a glass tube which passed through the rubber stopper, closing the end of the sound tube. The leads were taken from the same spool of wire, and were led in through the same tube parallel to the thermometer wires, and passed through the glass tube around which the thermometer wire was wound. The four wires were led to binding posts, and a plug switch-key, which was connected by wire cables to the Wheatstone's bridge in another part of the room.

Measurement of Wave-length. — The half wave length, in the standard tube was measured by placing a meter measure directly beneath the sound tube. Those in the comparison tube were measured by a bifilar index, so arranged that the nodes could easily be read, even while the large Dewar bulb contained liquid air. The position of the bifilar index was determined by two meter measure, parallel to the comparison tube, along which the bifilar index was caused to slide.

Calibration of Copper Thermometer — The copper thermometer was calibrated by determining the resistance at three fixed points. The points selected were: the temperature of steam, over boiling water, at known pressure, the temperature of melting ice, and the

temperature of liquid air of known vapor composition. Since the temperature of liquid air could only be determined by its analysis and the use of Baly's curve, a number of determinations were made at this temperature, with air having varying compositions, and the curve was smoothed through these points. Table I. gives the result of the calibration of the copper thermometer, from which the calibration curve, Fig. 3, is drawn, each measurement being the mean result of several readings.

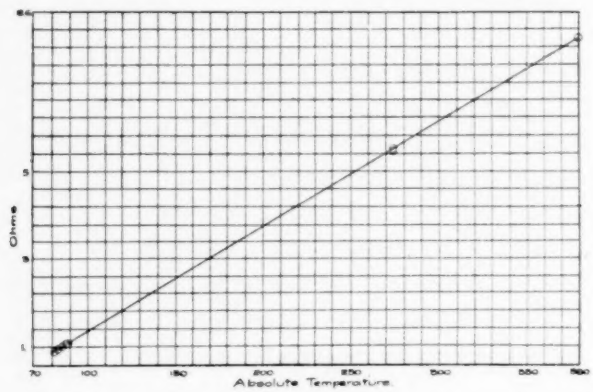

Fig. 3.

TABLE I.

No. of Readings Taken.	Resistance in Ohms.	Temperature Absolute.	Constant Temperature Substance.
10	8.019	372.9	Steam.
10	5.669	273.0	Ice.
10	5.664	273.0	Ice.
10	5.660	273.0	Ice.
10	3.698	193.0	Co_2 + ether.
10	.957	82.1	Liquid air.
10	.980	83.4	" "
8	1.007	84.8	" "
10	1.044	85.96	" "
10	1056	86.66	" "

Method of Combining the Results of the Measurements of the Half Wave-Lengths. — In the standard sound tube, where the adjustment of the distance between the end of the vibrating rod and adjustable cork could be obtained with very close approximation,

before the experiment began, and in which the cork dust could be redistributed by rotating the tube; well defined and perfectly uniform nodes could always be obtained. The positions of the nodes in either tube were measured successively to the nearest millimeter, and the readings were combined to obtain the mean value of a half wave-length.

Several methods of combining a series of measurements, in order to give a mean result, suggest themselves. If n be the number of nodes measured, and r be the number of any reading, (1) we may form the differences $r-(r+1)$ etc. to the difference $(r-n)$ giving $n-1$ differences, and take the mean of these differences; (2) we may form the differences $r-(n/2+r)$, etc. to the difference $(r-n)$ and take the sum of these differences divided by $(n/2)^2$ if n is even, and by $(n/2+1/2)(n/2-1/2)$, if n is odd; (3) we may form the differences $r-n$, $r-](n-(r-1))]$ etc., and divide the sum of these differences by $(n/2)^2$, if n is even and by $(n/2+1/2)(n/2-1/2)$ if n is odd. The first of these methods uses each half wave-length but once, and any errors in the measurements of the position of the nodes enter directly into the half wave-lengths. The second and third methods use respectively, the same number of half wave-lengths, *i. e.* $(n/2)^2$, if n is even, and $(n/2+1/2)(n/2-1/2)$, if n is odd. The only difference in the second and third methods being a difference in the distribution of the errors in the readings.

In measuring a series of half wave lengths, if the nodes are not all equally distinct, the readings of the individual nodes may be given weights; the resulting differences, between weighted nodes, will then carry with them the weights of the nodes. In measuring the half wave-lengths, in the comparison sound tube in the Dewar bulb, I found it convenient to use a combination of the second and third methods, using the third method without regard to the weights given each node, and using the second method to combine these nodes that had the highest weights assigned to them. The combination of these two methods gave very consistent results.

Determination of the Velocity of Sound in Air.

My first idea was to determine the velocity of sound in air, oxy-

gen, nitrogen, and possibly hydrogen, at the temperature of liquid air only, and at atmospheric pressure. The first determinations made in air, and oxygen, had this object in view.

Each determination was made in the following manner: When all parts of the apparatus were in place, and the cork dust uniformly distributed over the interior of the sound tubes, the sound tubes were filled with dry, carbon dioxide free air.

The Dewar bulb was then filled with liquid air as already described. As the air in the sound tube cooled, causing the pressure, as indicated by the manometer, to decrease, more air was allowed to flow into the sound tube. For the first determinations in air, the sound tubes were at atmospheric pressure. One observer followed the resistance of the thermometer, by means of the Wheatstone bridge and galvanometer; while the other observer manipulated the apparatus for filling the Dewar bulb, and at the proper time produced the sound wave. When the Dewar bulb was filled with liquid air, and the resistance thermometer indicated that the column of air in the sound tube had reached a steady temperature; the vibrating rod was bowed by rubbing it with a damp cloth. The length of the column of air between the two corks was adjusted, until good nodes were obtained in both sound tubes, the temperature, as indicated by the resistance thermometer, and by the mercury thermometer, near the standard tube, were noted; and the nodes were measured. I could usually obtain one series of measurements of the nodes in the comparison sound tube in the Dewar

TABLE II.

Velocity of Sound in Air at the Temperature of Liquid Air.

No. of Experiment.	R Resistance in Ohms.	θ Tempt. Absolute.	θ' Tempt. Absolute.	P cm. of Mercury.	$\frac{1}{2}\lambda$ cm.	$\frac{1}{2}\lambda'$ cm.	u' cm./sec.	u cm./sec.	u'' cm./sec.
1	1.810	117.5	295.1	—	1.9856	3.289	34,559	20,863	21,800
2	1.255	94.8	295.0	74.38	1.7744	3.2775	34,553	18,705	19,580
3	1.194	92.5	297.0	74.50	1.7362	3.2910	34,673	18,296	19,350
4	1.152	90.7	293.8	74.36	1.7261	3.2791	34,483	18,152	19,170
5	1.218	93.5	294.3	73.47	1.7305	3.2850	34,511	18,180	19,450
6	1.200	91.5	298.0	73.22	1.7306	5.304	34,733	18,193	19,250
7	1.185	92.0	292.4	—	1.738	5.233	34,400	18,493	19,380

bulb before the liquid evaporated. Other measurements were made at room temperature, and corrected for the expansion of the glass tube.

Table II. sets forth the results of seven measurements of the velocity of sound in air.

In Table II. the absolute temperature is denoted by θ, the wave length by λ, and the velocity of sound by u. The primes refer in each case to the temperature, wave lengths, and velocity, in the standard tube, while u'', in the last column, is the velocity calculated from the formula for the velocity of sound, assuming the Boyle-Gay-Lussac-Avogadro law to hold down to liquid air temperature. For the velocity of sound in dry air at 0° I have taken, in accordance with the International Congress of Physics, Paris, 1900, the value 33136 cm. per sec. u'' has been calculated from the formula

$$u = u_0 \sqrt{1 + at}.$$

λ and λ' are mean values of several measurements of the wave length, measured at the temperature θ and θ' respectively. The velocity u at the temperature θ was calculated from the relation.

$$u = u' \frac{\lambda}{\lambda'}.$$

The absolute temperature θ was taken from the calibration survey of the copper thermometer, while θ' was found by assuming the absolute temperature to have the value $-273°$.

Measurement of the Velocity of Sound in Oxygen.

The method for measuring the velocity of sound in oxygen was similar to that for the measurement of the velocity of sound in air. The oxygen used, having been obtained by the electrolysis of water, contained traces of hydrogen. In order to remove the hydrogen, the oxygen was liquefied in a bulb immersed in liquid air. The tube was then exhausted and the vapor from the liquid oxygen allowed to pass into it.

Under atmospheric pressure, the oxygen again liquefied in the sound tube. At somewhat reduced pressure, however, the oxygen could be maintained in its vapor form even at the temperature of liquid air. The sound wave was passed through this saturated

vapor of oxygen, and, when the adjustment had been secured, and good flutings obtained, the temperature and pressure were noted, as in the air determinations.

It was necessary to make the determination at less than atmospheric pressure. This required maintaining a partial vacuum in the sound tube at the temperature of liquid air. Several determinations were made; but, as a very slight variation in the temperature produced a marked change in the pressure at which the oxygen would remain in the vapor condition, the determinations were not all made at the same pressure. This introduced into the results two variables, pressure and temperature. The number of the results, at varying pressure, has not been sufficient to establish any real dependence upon pressure; but the irregularity of the results showed some variable factors other than temperature. The difficulty of maintaining even a partial vacuum, at this low temperature, may account for some of the results that are higher than might have been expected from the results with air. An error due to either of these possible variations would have resulted in the increase of the wave-length of sound in the oxygen.

Table III. exhibits the results of the determination of the velocity of sound in oxygen at the temperature of liquid air. The symbols are used in a similar manner as in Table II.

TABLE III.

Velocity of Sound in Oxygen at Temperature of Liquid Air.

No. of Experiments.	R' Resist. in Ohms.	θ Degrees Absolete.	θ' Degrees Absolute.	l' Pressure in cm. of Mercury.	$\frac{1}{2}\lambda$ Wave-Length in cm.	$\frac{1}{2}\lambda'$ Wave-Length in cm.	u Velocity in cm./sec.	u' Velocity in cm./sec.	u'' Theoretical Velocity in cm./sec.
1	1.227	94.0	295.1	60.5	1.6276	3.274	17,177	34,550	18,550
2	1.574	107.7	291.2	50.8	1.8009	3.267	18,913	34,310	19,860
3	1.094	88.4	292.8	48.35	1.6489	3.266	17,368	34,400	18,000
4	1.532	106.1	293.1	40.15	1.7546	3.271	18,468	34,430	19,710
5	1.100	89.0	291.8	34.9	1.684	3.2697	17,692	34,350	18,040
6	1.117	89.7	290.5	34.0	1.695	3.2785	17,724	34,280	18,110
7	1.127	90.0	291.5	49.94	1.6393	3.2357	17,392	34,320	18,130

RATIO OF THE SPECIFIC HEATS.

To determine the ratio of the specific heats, at the temperature of liquid air, we have to solve equation 8, placing for an ideal gas

$d(pv) = 0$, and for a gas which does not follow the laws of an ideal gas we must solve equation 8, giving its values for Clausius's or Van der Waals's characteristic equation.

In order to solve these equations, it became necessary to determine the density of the air, or gas, at the temperature at which the wave-length λ had been determined, relative to the density at the temperature at which λ' had been measured.

From the ratio of the specific heats γ it was possible also to find the value of the ratio of the internal to the translational energy of the molecule, from formula 14 or 16. This involved the determination of the variation of the density with pressure; that is, the evaluation of the differential coefficient at the temperature θ.

Determination of the Density of Air at the Temperature of Liquid Air.

Apparatus. — An air thermometer bulb of about thirty cubic centimeters capacity was connected by a capillary glass tube to a volumetric burette of 100 c.c. capacity, graduated to fifths of a c.c. The burette was attached to a levelling tube, by a meter or more of heavy rubber tubing; and the apparatus was partly filled with mercury. The apparatus is shown in Fig. 4. The graduated burette, A, was connected to the gas cylinder, through f, and to the air pump and manometer through g. C was connected to the levelling tube, B, by rubber tubing, with a pinch-cock at n. By lowering the levelling tube the mercury could be run out into B; the pinch-cock at n closed; then by closing the delivery tube, f, the apparatus could be exhausted through g and filled with the gas whose density was to be found by opening the delivery tube, f.

The Air Thermometer Bulb. — The air thermometer bulb, D, shown enlarged in Fig. 4 at D', was made by drawing down a piece of cylindrical glass tubing, and sealing to it the capillary tube l. The platinum thermometer was then sealed in, and the tube was sealed off, just below the end of the platinum thermometer.

The Platinum Resistance Thermometer. — The platinum thermometer was made of fine platinum wire, laced through holes drilled in a strip of mica, M. This fine platinum wire was soldered to heavier platinum leads, a and b, which supported the thermometer, and

which were led out, through the walls of the air thermometer, to the copper leads above. The dummy, *d*, a piece of platinum wire of the same diameter as the heavy platinum wire, was soldered to the two copper leads, at the same position with respect to the thermometer bulb as were the thermometer leads. The copper leads were led to binding posts in a hard rubber base. This rubber base was divided into halves and supported by clamping tightly to the capillary tube, *l*. The binding posts were then connected, by twin cables, to the same switch key as was used for the copper thermometer; this, in turn, being connected by cable to the bridge.

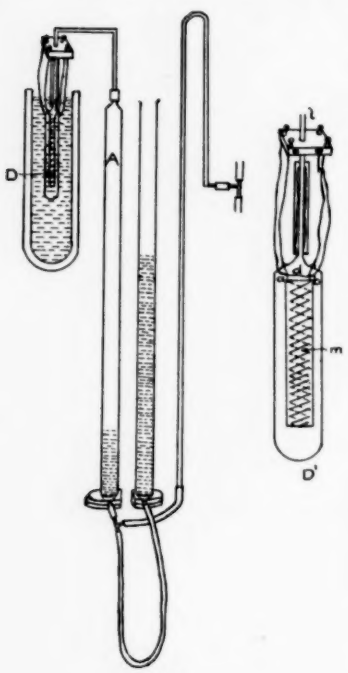

Fig. 4.

I made several of these air thermometer bulbs, and platinum thermometers, before one was obtained that gave consistent results. The chief difficulty was encountered in constructing a bulb that would stand liquid air temperature at the point where the heavy capillary tube began. This difficulty was obviated by covering some eight centimeters of the capillary with tinfoil. A piece of rubber tubing was fitted over this, to keep it in place, and to avoid any possibility of short circuiting the galvanometer by the platinum leads coming in contact with the tinfoil. The tinfoil acted as an equalizer of the temperature over the surface of the heavy capillary tube, and prevented all further breaks at this point. These facts are mentioned, mainly, to explain the cause for the use of several different air thermometer bulbs, during the determination of density. The different bulbs will be referred to by number.

DETERMINATION OF THE VOLUME OF THE AIR THERMOMETER BULB.

The volume of the bulb, fitted with the platinum thermometer, was determined by weighing it, filled with distilled water. The volume of the capillary tube was determined by means of a thread of mercury.

The data for determining the volume of the different bulbs, used in my determination of the density of air, and other gases, are exhibited in Table IV.

TABLE IV.

Volume of Bulbs.

1 No.	2 W grams.	3 W' grams.	4 T° C.	5 Temp. of Water °C.	6 V grams.	7 V cc.	8 V cc.	9 V cc.
1	149.703	245.832	21.4	21.5	96.129	96.490	96.186	95.626
2	125.051	170.652	18.8	17.5	45.6007	45.7154	45.42	45.20
3	129.085	159.4798	18.6	17.6	30.3948	30.472	30.172	30.018
2 A	128.106	164.356	22.1	22.1	36.250	36.370	36.070	35.900
5	132.922	162.344	16.5	18.5	29.422	29.517	29.207	29.057

Column 1 gives the number by which the bulb was known in the experiment; column 2 gives the weight of the bulb, when filled with dry air; column 3, the weight of the bulb, when filled with water; column 4 and 5 give the temperature at which the respective weighings were made; column 6, the apparent volume of the bulb and capillary tube, obtained by subtracting W from W'; column 7, the volume at 20° C.; column 8 gives the volume of the thermometer bulb less the volume of the capillary tube which was not at the liquid air temperature, and column 9, the volume of the thermometer bulb at −190° C.

CALIBRATION OF PLATINUM THERMOMETERS IN THE AIR THERMOMETER BULBS.

The platinum thermometers were calibrated, after they were sealed into the bulbs, by determining the resistance at the three points, *i. e.*, steam over boiling water, melting ice, and liquid air.

Table V gives the calibration of the thermometers. The calibration curve for one of the thermometers is exhibited in Fig. 5.

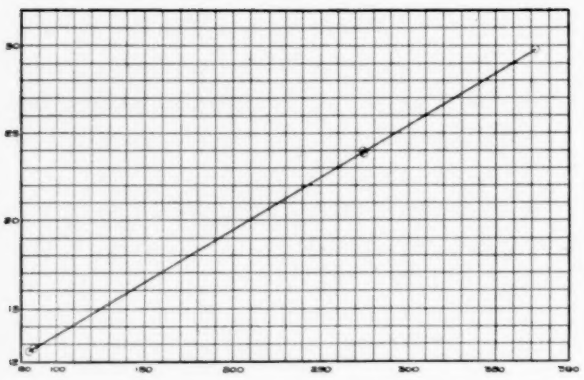

Fig. 5.

TABLE V.

Calibration of Thermometers in Air — Thermometer Bulbs.

Number.	Substance Used to Determine Fixed Point.	No. of Readings Taken.	Resistance of Therm. in Ohms.	Tempt. Absolute.
	Steam.	10	3.090	372.22
		10	3.000	372.24
	Ice.	10	2.219	273.00
1.		10	2.224	273.00
	Liquid air.	10	0.674	84.3
		10	0.684	87.0
		10	0.682	85.75
	Steam.	10	7.484	372.48
2.	Ice.	10	5.546	273.00
	Liquid air.	10	1.998	86.5
	Steam.	10	4.146	
3.	Ice.	10	3.117	273.00
	Liquid air.	[10	1.278	88.9
	Steam.	10	29.57	372.1
		10	29.81	372.1
5.	Ice.	10	23.90	273.0
		10	24.04	273.0
	Liquid air.		12.716	83.84
			12.908	85.77

Determination of the Relative Density of Air at the Temperature of Liquid Air.

Method. — After the apparatus had been well washed with dry air, it was exhausted; and the air was led in through *f*, having first passed through a cooling coil, placed in liquid air, which removed all traces of moisture or carbon-dioxide from the air. When the apparatus was filled with this pure air, the pinch cock at *n* was opened; and the mercury, flowing into the burette *A*, entrapped a certain volume of air. The mercury in *A* and *B* was now brought to the same level, and the barometer, which gave the pressure of the air in the apparatus, was read. The temperature of the air in the thermometer bulb was determined by the platinum thermometer, that in the burette by a mercury thermometer, suspended near *A*. The volume was then read on the burette. This gave the total volume of the gas enclosed. The air thermometer bulb, *D*, was immersed in liquid air. After the gas in the air thermometer bulb had attained a stationary temperature, as shown by the constancy of the resistance of the platinum thermometer, the mercury in the levelling tube was again brought to the same level as the mercury in the burette, and the temperatures of the gas in the air thermometer bulb, and that remaining in the burette, were determined by their respective thermometers. The pressure was again determined by means of the barometer. This gave sufficient data for the determination of the density of the air at the temperature indicated by the platinum thermometer, relative to the density of the air at room temperature.

To determine the relative density at any other than atmospheric pressure, the leveling tube was raised or lowered by any amount whatever within the limits of the burette; and the pressure difference was measured by a meter measure, placed behind the burette and levelling tube. The position of the mercury, in both the levelling tube and the burette, was determined by aluminum indices, made to slide over the glass tubes. These indices carried verniers, so that the scale on the meter measure could be read to tenths of millimeters. For low pressures the burette and bulb were filled, when at room temperature, at a pressure less than one atmosphere.

The burette was graduated to 0.2 cc., and could be estimated

to 0.02 cc. This gave an accuracy of two parts in a thousand, in the measurement of volume. The pressure could be measured to one part in a thousand. The temperature of gas in the burette could be read to tenths of a degree; and in the bulb, to the tenth of a degree from the calibration curve, which made the least reading for temperatures about one part in two thousand. The following tables exhibit the determinations of the relative densities of air, at various temperatures and pressures. Table VI. exhibits the den-

TABLE VI.

Relative Density of Air, at Various Temperatures, Pressure Constant. Bulb No. 5, *Pressure* 73.49 = p.

No. of Determination.	Bulb No.	Temperature.		Density. ρ
		$t°$ C.	$\theta°$ Absolute.	
1	1	20.6	83.8	3.661
2	3	17.8	82.5	3.656
3	3	17.9	83.5	3.656
4	3	17.9	83.5	3.655
5	3	17.9	83.5	3.655
6	2A	21.0	82.5	3.647
14	5	17.2	83.6	3.595
20	5	17.4	83.5	3.578
25	5	17.4	84.6	3.584

TABLE VII.

No. of Determination.	Temperature at Which Determinations were Made.		Density. ρ
	t	θ	
1	16.9	290.5	1.000
2	16.9	255.5	1.1432
3	16.9	238.5	1.2297
4	16.9	237.0	1.2382
5	16.9	235.2	1.2450
6	16.95	219.5	1.3490
7	17.00	197.0	1.5197
8	17.00	178.4	1.7312
9	17.00	290.0	1.000
10	17.00	221.0	1.3439
11	17.00	175.6	1.7478
12	17.00	147.6	2.1516
13	17.10	114.4	2.6699
20	17.40	83.5	3.5778

sity, at the temperature of liquid air and atmospheric pressure, relative to the density at room temperature at atmospheric pressure. Table VII. exhibits the density with varying temperature, pressure constant, and Table VIII. exhibits the same with varying pressure, temperature constant.

TABLE VIII.

Relative Density of Air at Various Pressures, Temperature Constant.

Number.	Tempt. at which Determinations were Made.		Pressure p. in cm. of Mercury.	Density. ρ
	t.	θ		
14	17.20	83.6	69.59	3.5950
15	17.40	83.5	61.09	3.5836
16	17.35	83.5	55.17	3.5768
17	17.35	83.5	50.95	3.5758
18	17.35	83.5	43.97	3.5512
19	17.40	83.5	39.84	3.5464
20	17.40	83.5	73.49	3.5778
25	17.40	84.6	73.49	3.5836
26	17.40	84.6	84.49	3.5821

DETERMINATION OF THE DENSITY OF OXYGEN.

The apparatus was the same as that for the density of air, and the method was identical. Oxygen liquefied at atmospheric pressure and liquid-air temperature. The point of liquefaction was determined by making a series of measurements at reduced pressures. As the pressure increased, the relative densities increased slightlyup to a certain point at which they increased much more rapidly and also became a function of the time that the bulb was kept at this low temperature. By plotting the relative densities as ordinates and the pressures as abscissa the graph made a sharp bend at the point of liquefaction.

The following tables give the relative densities of oxygen, as compared with oxygen at room temperature and atmospheric pressure. Table IX. gives the relative densities at varying pressure and constant or nearly constant temperature.

DETERMINATION OF THE RATIO OF THE SPECIFIC HEATS.

Having determined the density of air and oxygen, at the temperature of liquid air, it was only necessary to compute from these data

TABLE IX.

Density of Oxygen, Temperature Constant, under Varying Pressure.

Number of Determination.	Temperature at which Determinations were Made.		Pressure in cm. of Mercury p.	Density. ρ
	t	θ		
1	17.2	89.5	51.64	3.553
2	17.2	89.5	51.60	3.558
3	17.2	89.5	56.75	3.568
4	17.2	89.5	51.69	3.542
5	17.2	89.5	55.69	3.545
6	17.2	89.5	55.09	3.553
7	18.0	84.1	55.90	3.524
8	18.0	84.1	51.90	3.515
9	18.0	84.0	49.00	3.512
10	18.0	84.0	45.40	3.547
11	18.0	84.0	42.91	3.533
12	18.0	84.0	40.82	3.530
13	18.0	84.0	38.13	3.538
14	18.0	84.0	36.14	3.584
15	18.0	84.0	34.34	3.562

the density of air and oxygen, at the temperature at which the determination for the velocity of sound was made, relative to the density of air in the standard tube, in order to give sufficient data for the determination of the specific heats, at the temperature at which the velocity of sound was measured, from the formula,

$$v = v' \rho \left(\frac{\lambda}{\lambda'}\right)^2$$

TABLE X.

The Ratio of the Specific Heats in Air.

No.	P	θ	θ'	ρ	λ	λ'	γ
1		117.5	295.1	2.69	1.9856	3.2890	1.386
2	74.38	94.8	295.0	3.32	1.7744	3.2775	1.370
3	74.50	92.5	297.0	3.41	1.7362	3.2910	1.337
4	74.36	90.7	293.8	3.44	1.7261	3.2791	1.342
5	73.47	93.5	294.3	3.33	1.7305	3.2850	1.300
6	73.22	91.5	298.0	3.45	1.7306	3.3040	1.333
7		92.0	292.4	3.38	1.7380	3.2330	1.375

Table X. exhibits the results of the determination of the density, P, and the ratio of the specific heats γ. The value 1.408 was taken

as the ratio of the specific heat for air at normal temperature and pressure. The first column refers to the number of the experiment. The column P exhibits the pressure; θ and θ', the absolute temperature; λ and λ', the half wave-lengths; and γ the ratio of the specific heats.

Determination of the Ratio of the Internal to the External Energy of the Molecules.

Having determined the ratio of the specific heats, the ratio of the internal to the external energy of the molecule, is found from equations 10, 14 or 16, depending on whether we use the simple relation, in which $pv = R\theta$, the characteristic equation of Clausius, or that of Van der Waals.

In order to evaluate either equation 14 or 16, we must determine the value of the expression $1/p[\partial(pv)/\partial v]_\theta$.

To find the variation of pv relative to v, we measure the densities at different pressures. These measurements for air are exhibited in Table VIII.

Having assumed some arbitrary unit of pressure pv is found by multiplying the reciprocal of ρ by this pressure. The volume V, is computed by dividing the quantity pv by the actual pressure, P. The values of pv are plotted as ordinates, and the values of V, as abscissæ. The slope of the curve with the horizontal gives the value, $\partial(pv)/\partial v$; this, divided by the pressure at which the determination was made, gives the value of $1/p[(\partial pv)/\partial v]$

Table XI. gives the values of β for air.

Table XI.

Value of the Ratio of the Internal to the External Energy.

No.	γ	$\frac{1}{p}\frac{d(pv)}{dv}$	β_1	β_2	β_3
0	1.408	.0000	.614	.634	.634
1	1.385	.0012	.727	.732	.7295
2	1.370	.0012	.802	.807	.8045
3	1.337	.0012	.978	.983	.9805
4	1.342	.0012	.949	.954	.9515
5	1.300	.0012	1.222	1.227	1.2245
6	1.333	.0012	1.002	1.007	1.0045
7	1.375	.0012	.951	.956	.9535

Column β_1 is the value of the ratio of the internal to the external energy, deduced from the simple relation between γ and β given in equation 10; column β_2 is its value in accordance with Clausius' characteristic equation according to equation 14; and column β_3, its value according to Van der Waals' characteristic equation, from equation 16. Number 0 gives the value of β at normal temperature and pressure.

DETERMINATION OF THE VELOCITY OF SOUND AT TEMPERATURES BETWEEN THE TEMPERATURE OF THE ROOM AND THE TEMPERATURE OF LIQUID AIR.

Since the velocity of sound for both air and oxygen was found to be less than that calculated, on the assumption that the Boyle-Gay-Lussac-Avogadro law maintains down to the liquid air temperature, it was thought that it would be of interest to determine the velocity of sound at various temperatures up to the temperature of the room.

The apparatus, as set up for the measurement of the velocity of sound at liquid air temperature, was used for the determination at temperatures greater than that of liquid air. Liquid air was introduced into the Dewar bulb, as before, with the exception that, instead of completely filling the Dewar bulb only a small quantity of liquid air was introduced at a time. The sound tube in the Dewar bulb was then allowed to reach a minimum temperature, as indicated by the copper thermometer. It was found that at this point of minimum temperature the thermometer indicated that the temperature in the sound tube remained constant, or nearly constant, for three or four minutes; and a determination of the velocity of sound could easily be made in this time.

After making a determination in this manner more liquid air was introduced into the Dewar bulb; and a second determination was made at a lower temperature. From four to five determinations were made in this manner between room temperature and the temperature of liquid air. Two series of these determinations have been made in air and the results are exhibited in tables XII., and XIII.

After making the first series of measurements it occurred to me that, possibly, the temperature in the sound tube did not become

uniform throughout the tube. In order to avoid any possible error from this source two metal strips were introduced into the sound tube, completely lining the sound tube, with a metallic lining, except a narrow strip, above and below, through which the position of the nodes could be determined. The second set of measurements was made with this temperature equalizer in the sound tube.

TABLE XII.

Velocity of Sound in Air at Various Temperatures. (Set 1.)

No.	R	θ	$\frac{1}{2}\lambda'$	$\frac{1}{2}\lambda$	V_θ
1	6.145	294.0	3.2187	3.2265	34,400
2	4.133	212.6	3.233	2.7145	28,886
3	3.236? 2.936	175.8 163.0	3.233	2.3450	24,951
4	2.634? 2.334	151.4 138.4	3.233	2.1545	22,935
5	2.266	136.5	3.233	2.1070	22,419
6	1.185	92.0	3.233	1.738	18,493

TABLE XIII.

Velocity of Sound in Air at Various Temperatures. (Set 2.)

No.	R	θ	θ'	$\frac{1}{2}\lambda'$	$\frac{1}{2}\lambda$	V_θ
1	6.170	295.5	289.0	3.199	3.2220	34,470
2	6.082	292.0	290.7	3.206	3.1695	33,830
3	4.912	244.2	291.3	3.222	2.9092	30,892
4	3.005	166.8	291.3	3.222	2.3887	25,375
5	4.201	216.0	291.3	3.222	2.7550	29,261
6	3.033	167.5	291.3	3.222	2.3945	25,433
7	1.940	123.0	291.3	3.222	2.0405	21,673

TABLE XIV.

Velocity of Sound in Oxygen at Various Temperatures.

No.	R	θ	P	$\frac{1}{2}\lambda'$	$\frac{1}{2}\lambda$	$V\theta$
1	6.130	293.0	73.44	3.2205	3.0830	32,855
2	4.910	244.6	73.44	3.2357	2.6618	28,240
3	3.982	206.5	73.44	3.2357	2.4906	26,426
4	2.899	162.0	73.44	3.2357	2.1983	23,322
5	2.613	150.3	73.44	3.2357	2.1149	22,444
6	2.246	135.5	55.34	3.2357	1.9805	21,012
7	1.127	90.0	49.94	3.2357	1.6393	17,392

Table XIV. exhibits a set of measurements of the velocity of sound in oxygen at various temperatures between room temperature and the temperature of liquid air.

The determination of the velocity of sound in air and oxygen, as exhibited in Tables II., III., XII., XIII., and XIV. are exhibited graphically in Fig. 6.

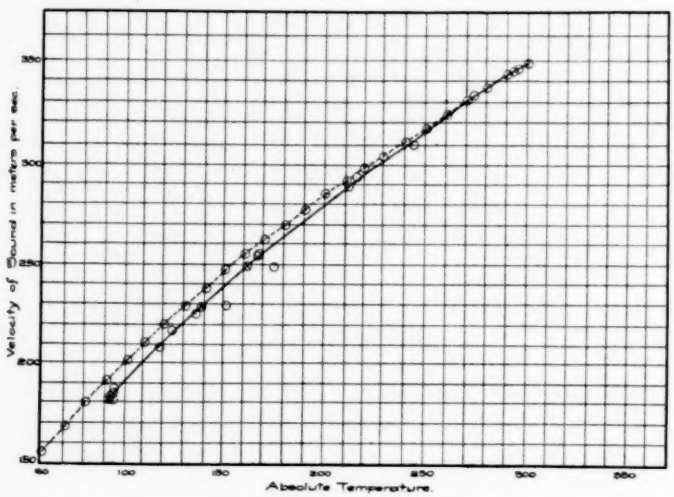

Fig. 6.

The velocity of sound is plotted as ordinates, and the absolute temperature as abscissæ. The curves are smoothed through the points. For comparison a curve (the broken line curve) which represents the velocity of sound, according to the formula,

$$u^2 = \gamma\, pv,$$

on the assumption that γ is constant and $pv = R\theta$ is plotted along with the curve plotted from the experiments.

CONCLUSION.

The results of the measurements of the velocity of sound show that the velocity of sound at low temperature does not follow the simple formula,

$$u^2 = \gamma\, pv.$$

If the pressure is constant, there remain two possible variables, γ and ρ. From the determination of the density at temperature between 0° and — 190°, it appears that the density, ρ, increases, relatively, more than would be consistent with $pv = R\theta$; but that it does not quite compensate for the decrease in the velocity of sound, and, hence, we find that the ratio of the specific heats decreases, somewhat, at the lower temperatures. At these low temperatures the gas is in the vapor condition. From the variation in γ it follows that there is a variation in the ratio of the internal to the external energy β.

The work detailed in this paper was undertaken during the summer of 1903. Through the courtesies of Professor Nichols and Dr. Shearer, the Physics Laboratory of Cornell University was placed at my disposal. The work was completed during the college year 1904 and 1905. I wish to take this opportunity of expressing my sincere thanks to professor Nichols and Dr. Shearer for many valuable suggestions during the progress of the work. I also wish to thank Dr. F. E. Kester for valuable assistance in making observations.

PHYSICAL LABORATORY,
CORNELL UNIVERSITY, June, 1905.

THE USE OF THE WEHNELT INTERRUPTER WITH THE RIGHI EXCITER FOR ELECTRIC WAVES.

BY A. D. COLE.

IN some earlier work with electric waves, generated by a Righi exciter and measured by a Klemenčič thermo-receiver,[1] it was found necessary to use a very sensitive galvanometer. Thus for most of the work a figure of merit of about 4.5×10^{-9}, in a 5-ohm galvanometer, was necessary.

Such sensitiveness in a galvanometer of resistance so low requires one to adjust its control magnet to produce a field so weak that the zero wanders badly and the vibration period is excessively long. This is a matter of serious consequence in these days when there is an electric railway or two in the near vicinity of nearly every physical laboratory. It is therefore desirable both for research and demonstration purposes to so increase the energy of the radiation as to allow the use of galvanometers of lower sensitiveness and shorter period.

As the integrated value of the energy is operative in receivers of the thermoelectric type, proportionately larger deflections are to be expected with an increase of frequency in the interrupter used with the induction coil. A test of this point was made several years ago by substituting a rotary break driven by an electric motor for the ordinary spring interrupter of the induction coil. Larger deflections were obtained in this way, but the action was quite irregular and the exciter soon became hot. So for the time the use of the ordinary interrupter was resumed.

Later, however, a plan for the constant circulation of oil through the exciter which had been adopted, gave promise of avoiding the troublesome heating effects of a high-frequency interrupter, and a new attempt was made. This time the Wehnelt interrupter was

[1] Cole, A. D., Wied. Ann., 57, 298, 1896. PHYS. REV., 7, 225, 1898; 20, 268, 1905.

tried and was found to answer the purpose admirably. A 4.5-ohm galvanometer whose figure of merit was 2.3×10^{-9} gave deflections of about 800 mm. with the scale 1 meter distant. The control magnet was moved until the figure of merit was only 4×10^{-8}, and its double vibration period only 2.7 seconds. Now the sensitiveness was only about one-twentieth what it had been before and yet the deflections are sufficiently large. The zero was steady (even with cars running every minute on the neighboring electric line). Further the greatly diminished period made it possible to obtain series of readings in a fraction of the time necessary before.

Considerable difficulty was found however at first in obtaining a form of interrupter which would remain in fairly constant condition for even an hour or two of steady use. The acid cell and the lead electrode were made of large size so that the interrupter would run cool, but the small electrode gave trouble. When made of platinum wire sealed into a glass tube it usually cracked out after a short time. The use of soft enamel glass for sealing in the wire reduced the difficulty, but did not remove it. An electrode which would not break was obtained by using several small platinum wires a millimeter or so apart sealed into the enameled tip. But these thin wires wore away too rapidly to be practical.

Various forms of adjustable electrode were tried. In these a single heavy platinum wire 1.4 mm. in diameter was used as an electrode. This was mounted in a surrounding glass tube of about 3.5 mm. internal diameter and projected through a plug or plate of insulating material at the lower end. Okonite insulation and rubber were first used for the plug, but they quickly softened by the heat and became disintegrated. Small discs cut from the bowl of a clay pipe were but little more durable. Compressed asbestos and a hard kind of talc were tried and rejected. Finally good results were obtained by using small discs cut out of a piece of close-grained slate. A small hole through the center of the disc allowed the platinum tip to be adjusted in length. This disc (see *a*, Fig. 1) was cemented to the end of the glass tube. The platinum wire was about 8 cm. long and soldered at *b* to one primary terminal of an induction coil. The cork shown at *c*, at the top of the glass tube, kept the tip from changing in length after it was suitably adjusted.

The tube usually projected down about 2 cm. into the weak acid of the cell. Details of construction are shown in the figure. The round battery jar of glass, 21 cm. high and 15 cm. in diameter, was filled to a depth of 13 cm with dilute sulphuric acid of specific gravity 1.05. One electrode was formed of a strip of lead 50 cm. long and 9 cm. wide, bent up into a U-shape so as to present about 720 sq. cm. of surface to the acid. The other electrode consisted of the 1.4 mm. platinum wire, projecting usually 4 mm. Acid so weak was chosen because the frequent removal and adjustment of the platinum tip was thus made more convenient and the small increase of resistance was unimportant since it was found best to use additional resistance in series with the cell for control.

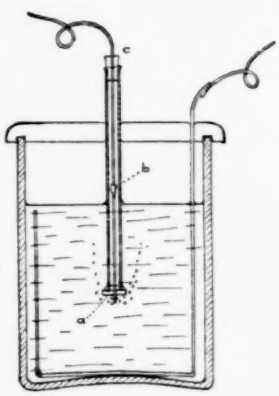

Fig. 1.

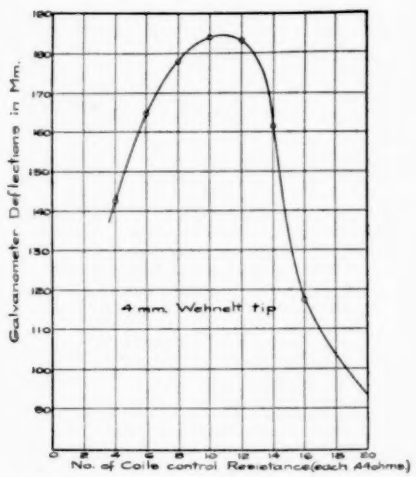

Fig. 2.

Alternating current of 110 volts was used for exciting this Wehnelt interrupter. When no resistance was used in series with it there was excessive heating, both of the interrupter and of the Righi exciter. This gave trouble in each by the occasional melting of the cement used in holding the parts together. The same trouble was experienced when the platinum wire of the Wehnelt tip was allowed to project more than about 6 mm. from the tube in which it was mounted. Of course direct current could be used equally well for exciting the Wehnelt interrupter.

The amount of control resistance which it is desirable to use in the primary circuit depends somewhat upon the length of tip used in the interrupter. Thus with 7.1 ohms of control resistance in the primary circuit, a 2 mm. tip gave better deflections than one of 10 mm., but with 4.4 ohms a 10 mm. tip gave the better effect. On the whole the best results were secured with the tip about 4 mm. long and with about 5 ohms of control resistance. The amount of effect caused by varying the resistance step by step with a fixed length of tip (4 mm.) is shown in Fig. 2.

The regularity of action that can be secured with the conditions right is illustrated by the following series of ten deflections taken in succession :

mm.	mm.
334	345
331	347
331	355
345	347
332	347

In this case 4.4 ohms of control resistance was used with the Wehnelt tip 10 mm. long.

The circulation of oil through the Righi exciter is a practically important feature, as it keeps the apparatus cool and prevents the irregularities in action which excessive carbonization of the oil would cause. This flow of oil is secured by drawing out the end of a long thistle-tube to a fine point and bending the tube to such a shape that it can be firmly held by a clamp to the upper edge of the frame of the parabolic mirror in which the exciter is mounted. It was adjusted to deliver about 30 drops of oil per minute to the vicinity of the central spark-gap. This construction is shown in Fig. 3. A constant-level overflow tube came up through the small oil chamber of the exciter to a height slightly above that of the spark-gap. After trying vaseline oil, olive

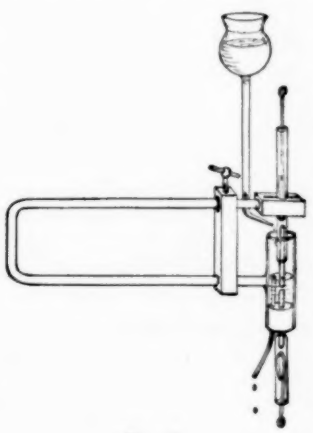

Fig. 3.

oil and lubricating oils of various degrees of viscosity, I have settled down upon ordinary refined petroleum (kerosene) as best adapted for use in exciters of short wave length.[1]

A Caldwell interrupter consisting of two vessels of acid with a small hole in a slate diaphragm between them was also tried, but proved inferior to the Wehnelt interrupter above described.

APPLICATION TO DEMONSTRATION PURPOSES.

This reduction of the necessary galvanometer sensitiveness to about one-twentieth of its former value makes the thermoelectric receiver for electric waves thoroughly practical for demonstration purposes, and its strictly quantitative character gives it an advantage over other indicators in use. The want of a suitable apparatus for demonstrating the optical analogies of electromagnetic radiation in class and lecture room has been seriously felt by many and the hope of securing it was the primary object of these experiments. Hertz's classic apparatus, so much used in Germany, is too cumbrous, its sensitiveness is small and its indications can be seen by only a few persons at once. The introduction of the coherer enabled Bose,[2] Lodge,[3] and others to reduce the apparatus to convenient dimensions, but this receiver is not strictly quantitative and it is very sensitive to all sorts of disturbing influences. The use of "copper hats" and small diaphragms to shield the coherer from side disturbances undoubtedly change materially the character of the radiation which is being studied. The mirror-strip receiver of Righi is obviously unsuited to demonstration purposes.[4]

The apparatus here described has been found as the result of

[1] To make an actual test of the magnitude of the results obtained with this Wehnelt interrupter in comparison with those secured with the ordinary type, two series of readings were taken, all the conditions being the same in each except that in the first the vibrator furnished with the induction coil was used (a Ritchie 4-inch coil with vibrating hammer interrupter) while in the second the Wehnelt interrupter was employed in its place. The mean galvanometer deflection in the first series was 9.2 mm. but in the second 96.1, or 10.4 times as much with the Wehnelt as with the ordinary interrupter. (Footnote added April 2, 1906.)

[2] Bose, J. C., Proc. Roy. Soc., 160, 167.

[3] Lodge, O. J., The Work of Hertz, etc., p. 32.

[4] Righi, A., Mem. R. Ac. Bologna, 1894. Phys. Soc. Abstracts, 1897, No. 196.

considerable experience to be well suited for demonstrations and I will close with a few illustrations of its usefulness in this direction.

A wave-length of 15 cm. has proved to be well adapted for demonstration. To secure this the total length of the two cylinders in the exciter should be 5 cm. and in the receiver 6 cm. The receiver is shown in Fig. 4. It is a special thermo-couple described more fully in PHYSICAL REVIEW, Vol. 20, p. 268. For most purposes both exciter and receiver are mounted at the foci of parabolic mirrors of sheet metal having an aperture about 35 × 35 cm., and placed from one to two meters apart.

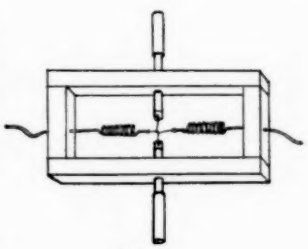

Fig. 4.

I have found it convenient to mount the receiver at the end of a revolving arm about 75 cm. long, swinging in the horizontal plane about a pivot half-way between exciter and receiver. This arrangement constitutes a spectrometer for electric waves.

All the experiments described in the culminating paper of Hertz's classical '87–'88 series[5] (and many others) are readily shown with this apparatus. For a plane reflector a sheet of metal 15 cm. square suffices, but it is better to use one 30 cm. square. A sheet of tin foil allows several per cent. of the radiation to pass through it, while ordinary sheet zinc transmits none. For a prism I use one of 30° angle and having faces 30 cm. square, cast in a wooden form with melted resin. Also a hollow one of glass plates filled with oil. My polarization grating is made of strips of tinfoil 2 mm. wide, 1 cm. apart on a sheet of cardboard. I am making a diffraction grating of tinfoil strips 12 cm. wide with 12 cm. spaces between them. For diffraction experiments I remove the parabolic mirror from behind the linear receiver and still have reasonably large deflections. (The multiplying power of the mirror is 6 to 8 fold.) For a concentrating lens I use an ordinary 5-pint acid bottle filled with benzine. Kerosene or gasoline will answer, but neither seems to be as good as benzine. Placing such a lens before a receiver will increase the radiation received about 5-fold. Stationary waves

[5] Hertz, H., Wied. Ann., 36, 769, '88.

are formed readily in several ways, as by perpendicular reflection from a plane mirror behind the receiver, or by a similar mirror behind the exciter, or by Boltzmann's two-mirror method.[6] I have given examples of curves showing the maxima and minima obtained by the first two methods in an earlier paper.[7] I add in Fig. 5 an example of the first method named.

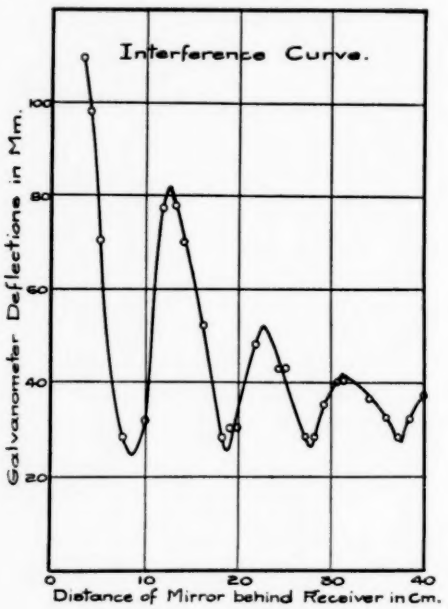

Fig. 5.

These illustrations suffice to show some of the uses of the apparatus and the results that can be obtained. Taking into consideration its comparative handiness, the ease with which its indications can be projected before a large class and the quantitative character of the results obtained, it seems to be adapted for demonstration purposes as well as for research.

PHYSICAL LABORATORY, OHIO STATE UNIVERSITY,
January 1, 1906.

[6] Boltzmann, L., Wied. Ann., 40, 399, 1890.
[7] Cole, A. D., PHYS. REV., 20, 270, 1905.

ON THE MEASUREMENT OF THE REFRACTIVE INDEX BY THE INTERFEROMETER.

By C. A. Proctor.

IN the paper published under the above title in the June number of this journal the authors seem to have lost sight of a source of serious error. When the two paths of light in any interference apparatus are made unsymmetrical by the introduction of a lamina of a dispersive medium into one of them, the central or achromatic fringe of the white light system no longer represents zero path-difference between the interfering rays but is determined by the condition

$$\frac{d\varphi}{d\lambda} = 0,$$

where φ is the phase difference between two "congruent" rays, and λ the wave length. This causes an increase in the apparent shift due to the introduction of the lamina, and the value of the index obtained from the fringe-count will be too large. This fact was first pointed out I believe by Sir. G. G. Stokes. The theory may be found developed in detail in Preston's "Theory of Light" and other standard works. The error due to this amounts to about 1.3 per cent. in the case of crown glass and increases with the dispersive power of the medium. It is therefore much too large to be disregarded.

It might be noted also that the method employed in measuring the thickness of the laminæ is subject to a theoretical error due to the relative phase change introduced between two rays which have undergone reflection at a glass-silver and an air-silver surface respectively. This error would however be very small if the lamina were of any considerable thickness, and could in any case be easily allowed for.

University of Missouri,
June 27, 1906.

PROCEEDINGS

OF THE

AMERICAN PHYSICAL SOCIETY.

Coefficient of Linear Expansion at Low Temperatures.[1]

By H. G. Dorsey.

In all previously recorded methods of determining the coefficient of linear expansion below about 200° absolute temperature it has been necessary to apply more or less corrections to get the final results or else there has been a possible doubt about the method.

In the present experiment there are no corrections.

The specimen to be examined, a hollow cylinder about a centimeter in length, is placed in a small steel box closed at the top with a glass window and is then carried through the range of temperature from 373° down to 93° absolute. Heat is supplied by a Bunsen burner, or extracted from the box by liquid air. Heating or cooling is so perfectly under control that the temperature may be maintained practically constant at any point between these limits.

The expansion or contraction of the specimen is measured by interference bands of green mercury light. The cylinder rests upon a plate of black glass and supports a plane wedge of glass. Since interference takes place by reflection from the top of the black glass and the bottom of the wedge no compensation is necessary. The air is exhausted from the box so there is no change in wave-length. Consequently each black band counted means an actual change in the length of the specimen of one half the wave-length of the light used.

Temperatures are measured by a copper resistance thermometer which has shown no change in its ice temperature resistance after a year's use. It was calibrated in steam, ice, liquid oxygen and liquid nitrogen.

The coefficient of expansion is measured for temperature ranges of 20°. This short interval brings out peculiarities which have been but little emphasized by previous workers.

Four substances have been studied, principally at the lower temperatures.

[1] Abstract of a paper presented at the Ithaca meeting of the Physical Society, June 29–July 3, 1906.

The coefficient curve for gold between 93° and 293° has eight points of inflection; maxima occuring at 122°, 198°, 218° and 255° and minima at 142°, 207°, 234° and 272°. The most sudden change is from 0.0000166 at 208° to 0.0000144 at 234°. To check the values at these sudden changes temperature intervals were made 10° as well as 20°. The mean coefficient between 93° and 273° is 0.0000131.

Copper gives a fairly regular curve, dropping from 0.0000162 at 283° to 0.0000104 at 103°. The curve has a slight convexity towards the temperature axis between 170° and 220° while the rest is concave, but there are no points of inflexion. The mean coefficient between 93° and 273° is 0.0000142.

Silver has maxima at 203° and 262° and a minimum at 226°; otherwise a general decrease from 0.0000199 at 283° to 0.0000139 at 103°. Mean coefficient between 93° and 273° is 0.0000166.

Flint glass (tubing) has a principal maximum at 223° and a slight one at 344°. At four places the coefficient remains nearly constant: 170° to 200° it is about 0.0000071, 240° to 260°, 0.0000083, 280° to 300°, 0.0000091, 340° to 374° 0.0000099. Value at 103° is 0.00000505. Mean coefficient between 93° and 373° is 0.00000778.

The work will be extended to a large number of substances including alloys.

The Temperature of the Moon.[1]

By Wm. W. Coblentz.

FROM his Mount Whitney observations Langley concluded that the soil of an airless planet at the moon's distance from the sun would have a temperature not greatly above — 225° C. His later observations on the radiation from the moon led him to conclude that the temperature of the sunward surface of the moon is about 0° C. His radiation curves rise from a zero value at 6 to 7 μ to a maximum at 8.6 μ, with a smaller maximum at 10 μ. He found also direct radiation from the sun in this region. The observations on the radiation from the moon during an eclipse show a drop of intensity from a maximum to a minimum in about 1.5 hours, which would seem to indicate that the observed radiation is reflected from the sun since the lapse of time is so short.

The writer has examined the infra-red reflecting power of a series of minerals including quartz, feldspars, amphiboles and micas, which are the chief constituents of the earth's crust, and has found the reflecting power to be practically zero for the region of the spectrum up to 8 μ,

[1] Abstract of a paper presented at the Ithaca meeting of the Physical Society, June 29–July 3, 1906.

followed by bands of metallic reflection from 8.5 to 10 μ. In this latter region the mean reflecting power is from 40 to 50 per cent. for the silicates, while pure quartz has a reflecting power of 90 per cent.

It follows, therefore, that the earth and the moon may be considered selectively reflecting surfaces with a band of metallic reflection from 8.5 to 10 μ. The radiation from the sun is so much more intense than that of the moon that, with a reflecting power of 50 to 90 p.c. at 8 to 10 μ, one would expect the amount of reflected energy to be, at all times, greater than the direct radiation from the moon.

The results as a whole show that it is just as reasonable to consider the radiation from the moon at 8.5 to 10 μ to be reflected from the sun as to consider it due to a temporary rise in temperature of the sunward side of the lunar surface, and Langley's first conclusions of a lunar temperature of — 225° C. again obtains.

WASHINGTON, D. C.,
June 15, 1906.

INFRA-RED REFLECTION SPECTRA.[1]

BY WM. W. COBLENTZ.

THE examination includes metals (*e. g.*, Zn, Cd, Pd) and sulphides of metals. The reflecting power of the sulphides of Zn, Fe and Sb are worthy of notice. They are known for their metallic lustre, especially stibnite, Sb_2S_3. The reflecting power of pyrite, FeS_2, rises from 20 per cent. at 1 μ to 35 per cent. at 14 μ and in its general trend is similar to that of iron. Sphalerite, ZnS, reflects only about 8 per cent. throughout the spectrum to 14 μ. Stibnite is the most extraordinary in having a reflecting power of about 37 per cent., to 12 μ. A section 0.4 mm. thick transmitted, on an average, about 46 per cent. throughout the spectrum, while a section 4.9 mm. thick transmitted 43 per cent., which after correcting for surface reflection indicates that the true absorption is only some 2 or 3 per cent. Such a great transparency has been found in but few substances except gases.

The refractive index of stibnite found by Drude for sodium light is $N = 4.49$ (5.17). The value calculated from the well known Fresnel formula, using the reflecting power for the region from 4 to 8 μ, gives a value of $N = 4.72$, and $N^2 = 16+$, while the observed dielectric constant is only about 10. Evidently there is something wrong in the assumption that stibnite is a homogeneous non-conducting medium.

[1] Abstract of a paper presented at the Ithaca meeting of the Physical Society, June 29–July 3, 1906.

Note on the Graphical Representation of Non-sinusoidal Alternating Currents.[1]

By Frederick Bedell.

THE current in an inductive circuit without iron lags ninety degrees behind an impressed sinusoidal E.M.F. and is itself sinusoidal. In the graphical representation, the current is represented by a line at right angles to the line which represents the E.M.F., these two lines determining a plane.

Iron causes a hysteresis loss and harmonics. The hysteresis loss introduces a power component of current of fundamental frequency, so that the total fundamental component of the current is advanced in phase, remaining in the plane of fundamental frequency.

The harmonics are wattless and hence are in quadrature to the E.M.F. It can be shown that graphically they are also in quadrature to the fundamental component of current, and accordingly are represented by a line at right angles to the plane of fundamental frequency.

The Calibration of Capillary Tubes.[1]

By Willard J. Fisher.

IN connection with researches on the viscosity of gases, H. Schultze[2] has described a method of calibrating tubes based on the combined use of long and short mercury threads. The following method, suggested by his, differs in the adoption of a graphical reduction, and a greater number of measurements. It is adaptable to calibrations for any purpose.

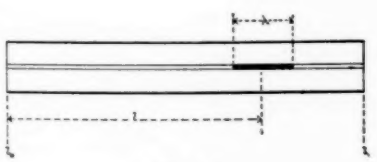

Fig. 1.

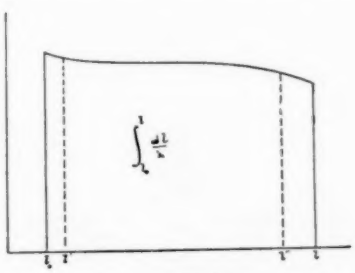

Fig. 2.

[1] Abstract of a paper presented at the Ithaca meeting of the Physical Society, June 29–July 3, 1906.

[2] H. Schultze, Ann. d. Physik, 5, p. 147, 1901.

I. *For Volumes.* — From any arbitrary position (in practice marked by etching the tube) designated by l_0, measure the position l of the center of a *short* mercury thread, also the length λ of the thread (Fig. 1). At the point l the cross-section of the tube is $Q = V/\lambda$, very nearly, in which V is the volume of the thread. Measured from l_0 the total volume of the tube to l is $v - v_0$. Also $dv = Qdl = V/\lambda dl$. If V is kept constant by temperature control,

$$v - v_0 = V\int_{l_0}^{l} \frac{dl}{\lambda}.$$

Plot a curve with $y = 1/\lambda$, $x = l$. Its area

$$= \int ydx = \int_{l_0}^{l} \frac{dl}{\lambda} = \frac{v - v_0}{V}.$$

If the cross-section of the tube is small the volume of the short thread cannot be computed accurately from weighings, since the container must be of greater mass than the mercury itself. Hence a more accurate way to find V is to find the total volume of the tube by a *long* thread between two points l' and l''. Call this volume $v'' - v'$. Then

$$V = \frac{v'' - v'}{\int_{l'}^{l''} \frac{dl}{\lambda}},$$

and is determined in terms of known quantities.

For thermometers the integral

$$\int_{l_0}^{l} \frac{dl}{\lambda}$$

can be taken between the limits given by the steam and ice points, and its value equal to 100° ; then the intervening points may be interpolated.

II. *Viscosity Tubes.* — The current of fluid is proportional to r^4/l, for uniform tubes, therefore to $1/(l/r^4)$. For nonuniform tubes put the sum of the quantities l/r^4 equal to F, which is then proportional to the resistance experienced by the fluid ;

$$F = \Sigma \frac{l}{r^4} = \int \frac{dl}{r^4}.$$

Assuming the cross-section circular, $Q = \pi r^2 = V/\lambda$

$$\therefore \frac{1}{r^4} = \frac{\pi^2}{V^2}\lambda^2$$

and between the limits l_0 and l_1,

$$F_1 - F_0 = \frac{\pi^2}{V^2}\int_{l_0}^{l_1} \lambda^2 dl.$$

Plot a curve with $y = \lambda^2$ and $x = l$. V is known from the curve for volumes; then $F_1 - F_0 = \pi^2/V^2$ into the area under the curve between the limits l_1 and l_0.

This theory can evidently be extended to calibration for other functions of the radius.

In practice it is advisable to run the thread through the tube more than once, as the points on the curve of one run may not exactly coincide with those for another.

The Production of Ozone by a Photo-Electric Current in Oxygen.[1]

By Harry S. Hower.

ULTRA-VIOLET light was allowed to fall upon a polished platinum knob in an oxygen atmosphere.

The knob being changed to about — 1,500 volts, the discharge current from it to the earthed wires opposite it was measured by a galvanometer in series. The ozone produced was detected by its effect on a piece of platinum foil. The foil changes its position in the voltaic series when it absorbs O_3.

Note on Electric Double Refraction in Carbon-Disulphide at Higher Temperatures.[1]

By L. B. Morse.

AT a former meeting of the Physical Society the late D. B. Brace reported that he had not been able to observe the "Kerr effect" in air. A beam of plane polarized light showed no trace of ellipticity after passing back and forth several times in the air between long charged plates.

Schmidt's curve, for the effect of temperature on the Kerr effect in CS_2, shows a more rapid rate of decrease of the effect with increase in temperature, at the higher than at the lower temperatures at which he worked.

At the time this work was begun an apparatus was being constructed in which ordinary gases, gaseous CS_2, etc., were to be tested under varying conditions of temperature and pressure.

In view of the above and other considerations it seemed important to

[1] Abstract of a paper presented at the Ithaca meeting of the Physical Society, June 29–July 3, 1906.

see whether or not there was a great decrease in the effect at temperatures nearer the boiling point.

Next to the difficulty of keeping constant and uniform temperature throughout the tube the greatest obstacle to be overcome was to keep the ends of the tube containing the hot CS_2 closed with cover glasses free from strains.

Cover glasses (free from double refraction) were cemented with thin water glass over holes drilled in thick plate glass. The larger part were spoiled in drying. Two of the more perfect were held by springs firmly against the ground glass ends of the tube, and the joint wrapped with twine soaked with water glass.

Readings taken at 24.4° and 41.6° would fit nicely on an extension of Schmidt's curve — showing that there is no marked change in the rate of decrease of the effect.

Certainly there is nothing in the behavior of CS_2 under electric strain at this temperature which would lead one to expect the effect to be zero at the boiling point, or in the gaseous state.

ELECTRIC DOUBLE REFRACTION IN CARBON DISULPHIDE AT LOW POTENTIALS.[1]

BY L. B. MORSE.

WHAT is known as Kerr's law of double refraction (which has been verified by later investigators, using high potentials) may be written as follows:

$$\delta = B\frac{P^2}{a^2}l,$$

where B is a constant depending upon the dielectric, l the length and a the distance between the plates, P is the difference in potential between the plates. From PHYSICAL REVIEW (paper by Elmen), January, 1905, "there seems to be a decided variation from the above law for low potentials."

The variation of the points on the curve published was naturally rather large near the limit of sensibility. Knowing that several improvements had increased the sensibility of the apparatus an effort was made to determine the curve more accurately if possible. Readings could be taken at lower potentials, and with less error at the same potentials used in the above investigation.

[1] Abstract of a paper presented at the Ithaca meeting of the Physical Society, June 29–July 3, 1906.

In order to obtain more accurate results observations were repeated till the eye became strong enough to take readings for a whole curve from lowest to highest potentials in a very short time. The effect at the lowest potentials was measured first — then up to the higher, where the eye could read to a greater relative accuracy than at the lower potentials even when very tired.

After entirely changing the parts of the apparatus and readjusting, the same results (approximately) were obtained.

All the results obtained indicate that Kerr's law holds within the range of probable experimental error.

A large part of the increased sensibility is without doubt due to a direct-connected filter system, and the *better* nicol prisms used, and perhaps in part to the more symmetrical position of the "dividing line" used in matching.

The Production of Ozone by Becquerel Rays Passing Through Oxygen.[1]

By Harry S. Hower.

VARIOUS methods of detecting the cause of the ozone odor observed near radium capsules in air were unsuccessful. By the delicate method of the preceding abstract, the attempt was made to show which rays change O_2 to O_3. No effects were observed for β- and γ-rays. The α-rays gave a decided positive result.

[1] Abstract of a paper presented at the Ithaca meeting of the Physical Society, June 29–July 3, 1906.

NEW BOOKS.

Avogadro and Dalton — the Standing in Chemistry of their Hypotheses. By ANDREW N. MELDRUM, D.Sc. Pp. 113. Edinburgh, James Thin, 1906.

The book consists of two essays, dealing with the present standing and relative importance of the hypotheses of Avogadro and Dalton. The author, whose position is sometimes more that of an advocate than of a judge, holds that the present importance of the former is the greater. After careful and critical study of the original papers and the meanings of the hypotheses to the two great originators themselves, he concludes that Avogadro's molecule is practically that to which present-day chemists are accustomed; while Dalton's atoms were rather unfruitful conceptions, never resulted in the evolution of order from the chaos of chemical facts accumulated during the first half of the nineteenth century, and are now in reality superseded.

The first essay, of seven chapters, shows in order that Avogadro developed his hypothesis as an explanation of Gay-Lussac's law of the simple volume ratios which are observed in the combination and decomposition of gases — then not with special reference to the laws of Boyle and Charle, which is contrary to common impression; that the deduction of Avogadro's hypothesis from the kinetic theory of gases might well be reversed, and the usual assumption of the identity of thermal and kinetic equilibrium among the gas-molecules might as well or better be deduced from Avogadro's hypothesis; that molecular weights are always deduced on the assumption of the hypothesis, as applied to gases or liquids; that in the determination of atomic weights doubtful cases are always settled by reference to the hypothesis, or by analogies with cases where it can be applied, and that other hypotheses which conflict with it are invariably discarded; that the accepted ideas of valence and the molecular formulas of the elements H_1, O_2, O_3, P_5, etc., are due to this hypothesis; and that "purely chemical methods" — meaning, no doubt, purely *gravimetric* methods, — do not suffice even to determine the formula of water. To quote from p. 49, Avogadro's hypothesis "is the basis of our knowledge of molecules and radicals and atoms. Molecular weights being ascertained, aud then atomic weights, the periodic system becomes possible. The chief issues of the hypothesis are the molecular theory,

including the modern theory of solution, the atomic hypothesis, the doctrine of valency, and the periodic system — the last a perpetual source of speculation and experiment."

The second essay, also of seven chapters, begins with the attempt to extract from Dalton's words the meaning to himself of his hypothesis, and to compare with this the modern ideas of it; concluding that it was an essentially gravimetric explanation of matter, whose author not only did not accept Avogardo's view, in the form which Dalton himself had imagined it, but also rejected Gay-Lussac's experimentally derived law; and that modern Daltonians hold theories of a mixed gravimetric and volumetric nature very different from anything that Dalton himself believed. It proceeds with short expositions of the chemical systems of Berzelius, Gmelin and Gerhardt and Laurent, and states (p. 96) that in 1850 to 1860 at least five symbols for water were in use, HO, H_2O, O, H_2O_2, O_2; for acetic acid "there were formulæ in bewildering profusion"; and further, that according to his school, H_2O_2 might mean to the chemist water or hydrogen peroxide; C_4H_4, marsh-gas or acetylene; $C_4H_4O_4$, acetic acid or fumaric acid; CuCl, cuprous or cupric chloride. From this state of confusion we were delivered by the writings of Cannizzaro, published in 1858, forty-seven years after Avogadro's first paper, and made widely known in 1860. He had studied the accumulated data on the vapor-density of inorganic substances, and consistently applied to the subject the hypothesis of Avogadro, with the result of an atomic weight system which was universally accepted within four years, and has stood to this day with only such changes as would be brought about by refinements of analysis; where the old chemists differed about magnitudes of hundreds of per cent., the modern analyst makes changes of tenths or hundredths of per cent. In 1865 appeared Newlands' "Law of Octaves," the first of the forms of the periodic law, all of which, together with the doctrine of valence, were impossible under any former atomic weight system.

Whatever the actual meaning of Avogadro's hypothesis, it has fulfilled in high degree the condition which Mach has proposed, that acceptance of a doctrine is based upon the degree of economy of mental effort which it produces. And to the expression of the hope that others may obtain from the reading of Dr. Meldrum's little volume as much pleasure and profit as he, may the reviewer add that it might be well to follow the author's suggestion and make this hypothesis, so fundamental in chemistry, a chief prop instead of a mere side issue in the teaching of the Kinetic Theory?

W. J. FISHER.

Handbuch der Physik. WINKELMANN. (Zweite Auflage.) Leipzig, J. A. Barth. 1903.

In the second edition of Winkelmann's "Handbuch" the plan of the original work has been continued and greatly extended. The new edition is to consist of six volumes in place of the three volumes of the original work and a number of specialists have been added to the original staff of contributors. The subject of Sound, which in the first edition was treated, in Volume I., as a special part of mechanics, is given a volume to itself. The treatment of Electricity is extended to cover two volumes and Optics which was contained in Volume II. of the first edition is placed at the end of the work and appears as Volume VI.

The new edition is in no respects a revision of the first but consists of a series of entirely new monographs or treatises written by specialists in each department and bringing the subject up to date. Among the new contributors to the volumes on Electricity are Luther on "Electrolysis," Stark on "Electricity in Gases and Radioactivity," Gerdien on "Atmospheric Electricity," Cantor on "Voltaic Cells and Storage Batteries," and Duden on "Contact Electricity." In the first part of Volume VI. the work of Czapski on "Geometrical Optics" is supplemented by chapters on "Vision" and on "Photographic Objectives" written by von Rohr. There is likewise a chapter on "Apparatus for Projection" by Eppenstein.

The method followed in the production of Winkelmann's "Handbuch" gives it a place distinct from that of other general treatises on Physics. It does not supplant these but supplements them in an admirable manner and each edition supplements instead of supplanting the one that went before. At the time of writing this notice Volumes I., II., III., the second part of Volume V. and the second part of Volume VI. had not yet been received. A review of these will be given at a later day.